Path Planning above a Polyhedral Terrain

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***Abstract*— We consider the problem of path planning above a polyhedral terrain and present a new algorithm that for any $p \geq 1$, computes a ($c+\varepsilon$)-approximation to the L_p-shortest path above a polyhedral terrain in $O(\frac{n}{\varepsilon}\log n \log\log n)$ time and $O(n \log n)$ space, where n is the number of vertices of the terrain, and $c = 2^{(p-1)/p}$. This leads to an ε-approximation algorithm for the problem in L_1 metric, and a $(\sqrt{2}+\varepsilon)$-factor approximation algorithm in Euclidean space.**

I. Introduction

Computing shortest paths in geometric domains is a fundamental problem in robot motion planning. There is a large body of work in this area, a broad overview of which can be found in the survey by Mitchell [7].

The problem of computing a two-dimensional shortest path among a set of polygonal obstacles is widely studied, and there are algorithms [5] solving the problem in the Euclidean metric (or in any L_p metric, $p \geq 1$) in optimal running time $O(n \log n)$, where n is the total number of vertices of the polygonal obstacles.

In 3D space, the problem of computing a shortest path among a set of polyhedral obstacles is well-known to be NP-hard [2] even in L_1 metric. However, for several classes of obstacles, exact shortest paths can be computed efficiently. For example, if obstacles are vertical buildings (prisms) with a fixed number k of distinct heights, the Euclidean shortest path can be computed in $O(n^{6k-1})$ time [4]. Furthermore, if the obstacle is a single "polyhedral terrain", then the L_1-shortest path can be computed exactly in $O(n^3 \log n)$ time [8].

The first ε-approximation algorithm for the 3D Euclidean shortest path problem is given by Papadimitriou [9] with running time $O(n^4\varepsilon^{-2}(N + \log\frac{n}{\varepsilon})^2)$, where n is the total number of vertices of the polyhedral obstacles, and N is the maximum bit-length of the input integers. A different approach was taken by Clarkson [3] resulting in an algorithm which is faster when $n\varepsilon^3$ is large. Asano et al. [1] have slightly improved the running time of Papadimitriou's algorithm to $O(n^4\varepsilon^{-2}\log N)$.

A. Contribution of This Paper

In this paper, we consider the 3D shortest path problem in the presence of a polyhedral terrain (i.e. a polyhedral surface having at most one intersection point with any vertical line in z-direction), while distances are computed in general L_p metric ($p \geq 1$). The problem definition is as follows:

> *Given an n-vertex polyhedral terrain T and two points s and t on or above T, find the L_p-shortest path from s to t that fully stays on or above T.*

Each input coordinate is assumed to be represented using a rational number whose numerator and denominator are integers of bit-length at most N.

We present an efficient algorithm that computes a $(1+\varepsilon)$-approximation to the L_1-shortest path above a polyhedral terrain in $O(\frac{n}{\varepsilon}\log n \log\log n)$ time and $O(n)$ space. As mentioned earlier, there is an exact algorithm for the problem in L_1 metric requiring $O(n^3 \log n)$ time [8]. However, in practical applications the input terrain is an approximation of the reality. Therefore, exact solutions are often meaningless, and efficient approximation algorithms are usually preferred.

In general L_p metric, our algorithm computes a factor-$(2^{(p-1)/p} + \varepsilon)$ approximation to the L_p-shortest path above a polyhedral terrain in $O(\frac{n}{\varepsilon}\log n \log\log n)$ time and $O(n \log n)$ space. This gives a $(\sqrt{2}+\varepsilon)$-approximation algorithm for the problem in the Euclidean space. Furthermore, by picking ε appropriately, we will guarantee that the length of the approximate path is at most twice the length of the optimal path in any L_p metric, $p \geq 1$.

B. Paper Outline

This paper is organized as follows. In Section II, we introduce the notion of VHV-paths and specify the relationship between this kind of paths and the shortest paths above a polyhedral terrain. In Section III we give an algorithm that finds a crude approximation to the shortest VHV-paths. Using this algorithm and a pseudo approximation technique proposed by Asano et al. [1], we give our main algorithm in Section IV to approximate the shortest path above a polyhedral terrain. We conclude in Section V with an open question.

II. Preliminaries and Properties

For each $p \geq 1$, the L_p-distance between two points a and b in 3D space is defined as $(|x(a)-x(b)|^p + |y(a)-y(b)|^p +$

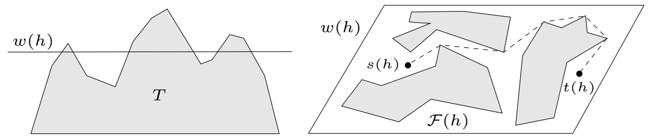

Fig. 1. A polyhedral terrain T and a plane $w(h)$ intersecting it. $w(h)$ is partitioned by T into a free region, $\mathcal{F}(h)$, and an obstacle region, shown in gray. The shortest path $\bar{\pi}(h)$ between $s(h)$ and $t(h)$ in the free region of $w(h)$ is shown by dashed lines.

$|z(a)-z(b)|^p)^{1/p}$. Special cases of the L_p metric include the L_1 metric (Manhattan metric) and the L_2 metric (Euclidean metric). The L_p-length of a polygonal path is the sum of the L_p-lengths of each segment of the path. The L_p-length of a path π is denoted by $\|\pi\|_p$. Throughout this paper, we assume that p is fixed. Therefore, we may suppress p in our notations. For example, we simply write $\|\pi\|$ instead of $\|\pi\|_p$.

Let T be a polyhedral terrain with n vertices, and let s and t be two points on or above T. We assume without loss of generality that $z(s) = 0$ and $z(t) \geq 0$. Let π_{opt} be the L_p-shortest path between s and t that fully stays on or above T. We note that π_{opt} is not necessarily unique.

Consider the plane $w(h) : z = h$. The intersection of T and $w(h)$ partitions $w(h)$ into a free region, $\mathcal{F}(h)$, and an obstacle region, $w(h) \setminus \mathcal{F}(h)$, as shown in Fig. 1. ($\mathcal{F}(h)$ consists of those points on $w(h)$ that lie on or above T). We denote by $s(h)$ and $t(h)$ the vertical projection of s and t on $w(h)$, respectively. Let $\bar{\pi}(h)$ be the L_p-shortest path from $s(h)$ to $t(h)$ that lies completely in $\mathcal{F}(h)$.

For $h \geq z(t)$, we construct a path from s to t above T as follows: we first move from s along a vertical segment to $s(h)$, then proceed from $s(h)$ to $t(h)$ along the planar path $\bar{\pi}(h)$, and finally descend from $t(h)$ along a vertical segment to t. We call such a vertical-horizontal-vertical path a *VHV-path with height* h and denote it by $\pi(h)$. Among all VHV-paths above T, we refer to the one with the minimum L_p-length as the *optimal VHV-path above* T and denote it by π^*. We note again that π^* is not necessarily unique.

Mitchell and Sharir [8] have observed that in L_1 metric, the optimal VHV-path and the L_1-shortest path above a polyhedral terrain have the same L_1-length. We can generalize this fact to any L_p metric as follows:

Observation 1: Let π^* be the optimal VHV-path, and π_{opt} be the L_p-shortest path above a polyhedral terrain. Then $\|\pi^*\| \leq 2^{(1-\frac{1}{p})}\|\pi_{\text{opt}}\|$.

Proof: Suppose that π_{opt} is composed of k segments $s_i = (a_i, b_i)$. We use x_i, y_i and z_i to refer to the length of the projection of s_i in the x-, y- and z-direction, respectively. Let c_i be the vertical projection of a_i on a horizontal plane passing through b_i, and define $\sigma_i = (b_i, c_i)$ and $h_i = (a_i, c_i)$ (see Fig. 2).

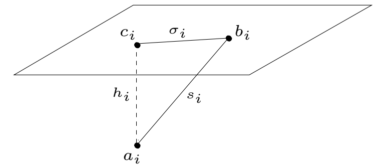

Fig. 2. The projection of segment $s_i = (a_i, b_i)$ on a horizontal plane passing through b_i.

Then

$$\frac{\|\sigma_i\| + \|h_i\|}{\|s_i\|} = \frac{(x_i^p + y_i^p)^{\frac{1}{p}} + z_i}{(x_i^p + y_i^p + z_i^p)^{\frac{1}{p}}} = \frac{\omega_i + z_i}{(\omega_i^p + z_i^p)^{\frac{1}{p}}}$$

where $\omega_i = (x_i^p + y_i^p)^{1/p}$. The above quotient is maximized when $\omega_i = z_i$, and hence, it is at most $\frac{2\omega_i}{\sqrt[p]{2\omega_i^p}} = 2^{(1-\frac{1}{p})}$.

Let h be the maximum z-coordinate of a point on π_{opt}. It is clear that $\|\pi^*\| \leq \|\pi(h)\|$. The path π_{opt} can be decomposed into two (possibly empty) z-monotone subpaths π^+ and π^-, where π^+ is ascending and π^- is descending in z-direction. Therefore, $\sum_i \|h_i\| = 2h - z(t)$. Furthermore, the vertical projections of σ_i's on $w(h)$ form a path from $s(h)$ to $t(h)$ that completely lies in the free region of $w(h)$. Therefore, $\sum_i \|\sigma_i\| \geq \bar{\pi}(h)$. Putting all together, we have

$$\begin{aligned}\|\pi(h)\| &= \|\bar{\pi}(h)\| + 2h - z(t) \\ &\leq \sum_{i=1}^{k} \|\sigma_i\| + \sum_{i=1}^{k} \|h_i\| \\ &\leq 2^{(1-\frac{1}{p})} \sum_{i=1}^{k} \|s_i\|\end{aligned}$$

which implies that $\|\pi^*\| \leq 2^{(1-\frac{1}{p})}\|\pi_{\text{opt}}\|$. □

By Observation 1, any algorithm that computes an ε-approximation to the optimal VHV-path above a polyhedral terrain T, provides a factor-$(2^{(p-1)/p} + \varepsilon)$ approximation to the L_p-shortest path above T in any L_p metric. The following observation will be a main ingredient of our ε-approximation algorithm in Section IV.

Observation 2: Let $h > h' > 0$. If $h - h' \leq \Delta/2$ then $\|\pi(h)\| \leq \|\pi(h')\| + \Delta$.

Proof: Let $L(h) = \|\bar{\pi}(h)\|$. The free region of $w(h)$, $\mathcal{F}(h)$, expands as h increases. Therefore, $L(h)$ is a decreasing function of h. It means that for $h > h'$, $L(h) - L(h') \leq 0$. Using the fact that $\|\pi(h)\| = L(h) + 2h - z(t)$ we get

$$\begin{aligned}\|\pi(h)\| - \|\pi(h')\| &= L(h) - L(h') + 2(h - h') \\ &\leq 2(h - h') \leq \Delta.\end{aligned}$$

□

III. Finding a Crude Approximation

In this section, we show how to efficiently find a crude approximation to the optimal VHV-path, π^*, above a polyhedral terrain. More precisely, we find a real value that approximates the length of π^* to within a multiplicative-factor of $O(n)$. This crude approximation will be then used in the next section to obtain an ε-approximation to π^*.

For $r > 0$, let $C_s(r)$ be a cube of side length $2r$ centered at s. We first prove the following simple lemma.

Lemma 1: Given a value $r > 0$, we can check in $O(n)$ time whether $C_s(r)$ contains a path from s to t that fully stays on or above T.

Proof: Let S be the top face of $C_s(r)$. To see if $C_s(r)$ contains a valid path from s to t, we just need to check if there is a path connecting $s(r)$ to $t(r)$ in $S \setminus T$. The intersection of T and the plane $w(r)$ forms a set of obstacles $O = \{O_1, O_2, \ldots\}$, where each obstacle is a simple polygon (we discard holes inside the obstacles). If T is stored in a proper data structures like a Doubly-Connected Edge List [10], we can obtain every O_i as a sorted list of its edges in total linear time. For each obstacle O_i, we then compute simple polygons resulted from $S \setminus O_i$, and use standard point location methods to check if $s(r)$ and $t(r)$ lie in the same polygon. This can be done in time linear to the size of O_i [10]. Thus, performing the check on all obstacles can be done in $O(n)$ overall time. □

The next lemma, shows how we can find a value r that approximates the length of the optimal VHV-path above T.

Lemma 2: Let π^* be the optimal VHV-path above T. We can find a value r such that $r \le \|\pi^*\| < 8nr$ in $O(n \log n)$ time and $O(n)$ space.

Proof: Let r^* be the smallest value for which $C_s(r^*)$ contains a valid path from s to t above T. Clearly, $r^* \le \|\pi^*\|$. If S is the top face of $C_s(r^*)$, then there is a path from $s(r^*)$ to $t(r^*)$ in $S \setminus T$ that consists of k segments of length at most $4r$. It is easy to observe that $k \le n - 1$. Therefore, $\|\bar{\pi}(r^*)\| \le 4(n-1)r^*$, and hence, $\|\pi^*\| \le \|\pi(r^*)\| \le 4(n-1)r^* + 2r^* < 4nr^*$.

Now we show how to find a 2-approximation of r^* in $O(n \log n)$ time. Let N be the maximum bit-length of the integers in the input coordinates. Then it is clear that $r^* \le 2^N$. Furthermore, we know that the shortest distance between any pair of points in this setting is 2^{-3N} (This is the distance between two parallel planes specified with integer coefficients of bit length at most N, and thus a 3×3 determinant of such integers [9]).

For every integer i, we define $r_i = 2^{i-3N-1}$. It is clear that $r_t \le r^* \le r_{t+1}$ for some $t \in [0, 4N]$. We use the idea of binary search to find t using at most $O(\log N)$ queries of the form "if $C_s(r_i)$ contains a valid path from s to t". According to Lemma 1, this requires $O(n \log N) = O(n \log n)$ overall time. By setting $r = r_t$, we simply have $r \le r^* \le 2r$, and hence $r \le \|\pi^*\| < 8nr$. □

IV. The ε-Approximation Algorithm

Let Π be the set of all VHV-paths between s and t that fully stay on or above T. For $R \ge 0$, we denote by Π_R the set of those paths in Π that lie completely in the half-space $z \le R$. In other words, Π_R is the set of those VHV-paths whose heights are restricted to be at most R. Let π^*_R be the L_p-shortest path in Π_R. For $R < R'$, we have $\Pi_R \subseteq \Pi_{R'} \subseteq \Pi$. Therefore,

$$\|\pi^*_R\| \;\ge\; \|\pi^*_{R'}\| \;\ge\; \|\pi^*\|$$

where π^* is the optimal path in Π. Furthermore, the following property holds true:

$$\|\pi^*\| \le R \quad\Longrightarrow\quad \|\pi^*_R\| = \|\pi^*\|.$$

According to this property, there is a direct correlation between the search radius parameter R and the length of the optimal path in Π_R. It enables us to use a pseudo approximation framework proposed by Asano et el. [1].

For $\varepsilon > 0$, a *pseudo approximation algorithm* for our problem computes a path $\pi(\varepsilon, R) \in \Pi_R$ such that

$$\|\pi(\varepsilon, R)\| \;\le\; \|\pi^*_R\| + \varepsilon R.$$

We call $\pi(\varepsilon, R)$ a *pseudo ε-approximation* to π^*_R. The following lemma provides an efficient pseudo approximation algorithm for our problem.

Lemma 3: For $R \ge 0$ and $\varepsilon > 0$, there is a pseudo approximation algorithm that computes a path $\pi(\varepsilon, R) \in \Pi_R$ such that $\|\pi(\varepsilon, R)\| \le \|\pi^*_R\| + \varepsilon R$ in $O(\frac{n}{\varepsilon} \log n)$ time and $O(n \log n)$ space.

Proof: The algorithm is straightforward: For each $1 \le i \le \lceil 2/\varepsilon \rceil$, we compute $\pi(h_i)$ at heights $h_i = i \times \varepsilon R/2$, and then, select the path with the minimum L_p-length among the computed paths as $\pi(\varepsilon, R)$. Let h^* be the maximum z-coordinate of a point on π^*_R, i.e. $\|\pi^*_R\| = \|\pi(h^*)\|$. Clearly, h^* falls in an interval $[h_{k-1}, h_k]$ for some $1 \le k \le \lceil 2/\varepsilon \rceil$. Since $h_k - h^* \le \varepsilon R/2$, Observation 2 implies that $\|\pi(h_k)\| \le \|\pi(h^*)\| + \varepsilon R$ and hence $\|\pi(\varepsilon, R)\| \le \|\pi^*_R\| + \varepsilon R$.

For the complexity, we recall that computing each $\pi(h_i)$ is equivalent to constructing a planar L_p-shortest path $\bar{\pi}(h_i)$, which can be accomplished in $O(n \log n)$ time and $O(n \log n)$ space [6]. (Indeed, we need just linear space in L_1-metric [5]). Computing $\lceil 2/\varepsilon \rceil$ such paths requires $O(\frac{n}{\varepsilon} \log n)$ total time. □

We call R a *low value* in case $\|\pi(\varepsilon, R)\| \ge R$, and a *high value* otherwise. Asano et al. have proved the following nice property.

Lemma 4: [1] For $\alpha > 0$, if R_l is a low value and R_h is a high value s.t. $R_h \le \alpha R_l$, then $\|\pi(\varepsilon, R_h)\| < (1 + \alpha \frac{\varepsilon}{1-\varepsilon}) \|\pi^*\|$.

By assuming $\alpha = 2$ and $\varepsilon \leq 1/2$, we always have $\alpha\varepsilon/(1-\varepsilon) \leq 4\varepsilon$. Using Lemma 4, one can therefore obtain a $(1+4\varepsilon)$-approximation to π^*, for any $\varepsilon \leq 1/2$, by simply finding a low value R_l and a high value R_h such that $R_h \leq 2R_l$. The following algorithm uses this fact to compute an ε-approximate VHV-path.

Algorithm 1 FIND ε-APPROXIMATE VHV-PATH

Find an r such that $r \leq \|\pi^*\| < 8nr$
Define $R_i = r2^i$, for all $i \geq 0$
$l \leftarrow 0$, $h \leftarrow \lceil \log_2 n \rceil + 4$
while $h - l > 1$ **do**
  $m \leftarrow \lceil (l+h)/2 \rceil$
  if $R_m \leq \|\pi(\varepsilon, R_m)\|$
    then $l \leftarrow m$
    else $h \leftarrow m$
Return $\pi(\varepsilon, R_h)$

Theorem 1: Algorithm 1 computes an ε-approximation to the optimal VHV-path above a polyhedral terrain in $O(\frac{n}{\varepsilon}\log n \log\log n)$ time and $O(n \log n)$ space.

Proof: The correctness of the algorithm easily follows from the following loop invariant: *At the beginning of each iteration, R_l is a low value and R_h is a high value.* Note that before the first iteration, $R_l = r \leq \|\pi^*\| \leq \|\pi(\varepsilon, R_l)\|$ and $R_h \geq 16nr$, thus $\|\pi(\varepsilon, R_h)\| \leq \|\pi^*_{R_h}\| + \varepsilon R_h = \|\pi^*\| + \varepsilon R_h < \frac{1}{2}R_h + \frac{1}{2}R_h = R_h$. Upon termination of the loop, we have $R_h = 2R_l$. Therefore, by Lemma 4 the output is a $(1+4\varepsilon)$-factor approximation to π^* for any $\varepsilon \leq 1/2$, and hence, the algorithm can be viewed as a $(1+\varepsilon')$-approximation algorithm for any $0 < \varepsilon' \leq 1/8$.

In each iteration of the loop, we need just one call to the pseudo approximation algorithm to verify whether R_m is a low value. The total number of calls to the pseudo approximation algorithm is thus $O(\log\log n)$. It immediately follows from Lemma 3 that the running time of our algorithm is $O(\frac{n}{\varepsilon}\log n \log\log n)$ and its space complexity is $O(n \log n)$. □

By Observation 1, any ε-approximation to the optimal VHV-path immediately gives a factor-$(2^{(p-1)/p} + \varepsilon)$ approximation to the L_p-shortest path above T. Theorem 1 is therefore equivalent to the following:

Theorem 2: For any $p \geq 1$, the L_p-shortest path above a polyhedral terrain can be approximated to within a factor of $2^{(p-1)/p} + \varepsilon$ using $O(\frac{n}{\varepsilon}\log n \log\log n)$ time and $O(n \log n)$ space.

Corollary 1: For any fixed $p \geq 1$, a 2-approximation to the L_p-shortest path above a polyhedral terrain can be obtained in $O(n \log n \log\log n)$ time and $O(n \log n)$ space.

Proof: It directly follows from Theorem 2 by picking $\varepsilon = 1/p$ and observing that $1/p \leq 2(1 - 2^{-1/p})$ for all $p \geq 1$. □

V. CONCLUSIONS

In the real world, aircrafts flying over a terrain usually follow a simple pattern: they first fly upwards to a certain height, then travel along a horizontal plane at that height to a point above the target, and finally descend to the target. In this paper, we showed how to efficiently approximate the optimal such vertical-horizontal-vertical path to within a multiplicative factor of $1 + \varepsilon$. This led to a simple and efficient algorithm for approximating the L_p-shortest paths above a polyhedral terrain to within a factor of $2^{(p-1)/p} + \varepsilon$. The running time of our algorithm is $O(\frac{n}{\varepsilon}\log n \log\log n)$ and its space complexity is $O(n \log n)$.

While there are several algorithms to approximate the Euclidean shortest path among a set of polyhedral obstacles, none of these algorithms is specialized for the case where the obstacle is a single polyhedral terrain. An interesting question is thus whether we can exploit properties of the polyhedral terrains to obtain more efficient ε-approximation algorithms for this especial case of 3D shortest path problem. The algorithm presented in Section IV gives a positive answer to this question in L_1 metric. For other L_p metrics ($p \geq 2$) the question remains open.

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