Graph Layouts via Layered Separators

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ABSTRACT. A k-queue layout of a graph consists of a total order of the vertices, and a partition of the edges into k sets such that no two edges that are in the same set are nested with respect to the vertex ordering. A k-track layout of a graph consists of a vertex k-colouring, and a total order of each vertex colour class, such that between each pair of colour classes no two edges cross. The queue-number (track-number) of a graph G, is the minimum k such that G has a k-queue (k-track) layout.

This paper proves that every n-vertex planar graph has track number and queue number at most $\mathcal{O}(\log n)$. This improves the result of Di Battista, Frati and Pach [*Foundations of Computer Science*, (FOCS '10), pp. 365–374] who proved the first sub-polynomial bounds on the queue number and track number of planar graphs. Specifically, they obtained $\mathcal{O}(\log^2 n)$ queue number and $\mathcal{O}(\log^8 n)$ track number bounds for planar graphs.

The result also implies that every planar graph has a 3D crossing-free grid drawing in $\mathcal{O}(n \log n)$ volume. The proof uses a non-standard type of graph separators.

1 Introduction

A *queue layout* of a graph consists of a total order of the vertices, and a partition of the edges into sets (called *queues*) such that no two edges that are in the same set are nested with respect to the vertex ordering. The minimum number of queues in a queue layout of a graph is its *queue-number*. Queue layouts have been introduced by Heath, Leighton, and Rosenberg [15, 19] and have been extensively studied since [1, 6, 7, 9, 10, 14, 15, 19, 23–28]. They have applications in parallel process scheduling, fault-tolerant processing, matrix computations, and sorting networks (see [23] for a survey). Queue layouts of directed acyclic graphs [2, 17, 18, 23] and posets [16, 23] have also been investigated.

The dual concept of a queue layout is a *stack layout*, introduced by Ollmann [22] and commonly called a *book embedding*. It is defined similarly, except that no two edges in the same set are allowed to cross with respect to the vertex ordering. *Stack number* (also known as *book thickness*) is known to be bounded for planar graphs [29], bounded genus graphs [21] and, most generally, all proper minor closed graph families [3, 4].

No such bounds are known for the queue number of these graph families. The question of Heath *et al.* [15, 19] on whether every planar graph has $\mathcal{O}(1)$ queue-number, and the more general question (since planar graphs have stack-number at most four [29]) of whether queue-number is bounded by stack-number remains open. Heath *et al.* [15, 19] conjectured that both of these questions have an affirmative answer. Until recently, the best known bound for the queue number of planar graphs was $\mathcal{O}(\sqrt{n})$. This upper bound

follows easily from the fact that planar graphs have pathwidth at most $\mathcal{O}(\sqrt{n})$. In a recent breakthrough [1], this queue number bound for planar graphs was reduced to $\mathcal{O}(\log^2 n)$, by Di Battista, Frati and Pach[1] [1]. The proof, however, is quite involved and long.

We improve the bound for the queue number of planar graphs to $\mathcal{O}(\log n)$. Pemmaraju [23] conjectured that planar graphs have $\mathcal{O}(\log n)$ queue-number. Thus the result answers this question in affirmative. He also conjecture that this is the correct lower bound. To date, however, the best known lower bound is a constant.

The proof is simple and it uses a special kind of graph separators. In particular, the main result states that every n-vertex graph that has such a separator (and it turns out that planar graphs do) has an $\mathcal{O}(\log n)$ queue number As such, the result may provide a tool for breaking the $\mathcal{O}(\sqrt{n})$ queue number bound for other graph families, such as graphs of bounded genus and other proper minor closed families of graphs.

One of the motivations for studying queue layouts is their connection with three-dimensional graph drawings in a grid of small volume. In particular, a *3D grid drawing* of a graph is a placement of the vertices at distinct points in $\mathbb{Z}^3$, such that the line-segments representing the edges are pairwise non-crossing. A 3D grid drawing that fits in an axis-aligned box with side lengths $X-1$, $Y-1$, and $Z-1$, is a $X \times Y \times Z$ drawing with volume $X \cdot Y \cdot Z$.

It has been established in [6, 7], that an n-vertex graph G has an $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ drawing, if and only if G has $\mathcal{O}(1)$ queue-number. Therefore, if a graph has a bounded queue number then it has a linear volume 3D grid drawing. One of the most extensively studied graph drawing questions is whether planar graphs have linear volume 3D grid drawings – the question is due to Felsner *et al.*[11]. Our results imply $\mathcal{O}(n \log n)$ bound, improving on the previous $\mathcal{O}(n \log^8 n)$ bound [1].

In the next section, we give precise statement of our result and introduce a tool used to obtain it. In Section 3 we prove the main result and then conclude with some open problems in Section 4.

2 Results and Tools

The main tool in proving our result is the following type of graph separators.

A *layering* of a graph G is a partition $V_0, V_1, \ldots, V_p$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $|i-j| \leq 1$. Each set V_i is called a *layer*. A *separation* of a graph G is a pair (G_1, G_2) of subgraphs of G, such that $G = G_1 \cup G_2$ and there is no edge of G between $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$.

A graph G has a *layered ℓ-separator* if for some fixed layering L of its vertices the following holds: For every subgraph $G' \subseteq G$ there is a separation (G_1', G_2') of G' such that each layer of L contains at most ℓ vertices in $V(G_1') \cap V(G_2')$, and both $V(G_1') - V(G_2')$ and $V(G_2') - V(G_1')$ contain at most $\frac{2}{3}|V(G')|$ vertices. Here the set $V(G_1') \cap V(G_2')$ is a (layered $\ell-$) *separator* of G'. Finally, if a graph G has a *layered ℓ-separator* for some fixed layering L, we say that G has (ℓ, L)–*separator*. Note that these separators do not necessarily have small

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[1]The original bound proved in this conference paper, [1], is $\mathcal{O}(\log^4 n)$. The bounds stated here are from the journal version that is under the submission.

order, in particular $V(G_1') \cap V(G_2')$ can have linear number of vertices of G'.

The notion of layered separators is not new. They were used implicitly, for example, in the famous proof, by Lipton and Tarjan [20], that planar graphs have a separator of order $\mathcal{O}(\sqrt{n})$. Specifically, consider a breath-first-search tree T of a graph G and the layering L defined by partitioning the vertices of G according to their distance to the root of T. Each edge that is not in T defines a unique cycle, called a T-*cycle*. One step in their proof was to show that any edge maximal planar graph has a T-cycle separator. A T-cycle contains at most two vertices from each layer of L. However, to jump from the existence of a T-cycle separator of G to the existence of a layered 2-separator of G requires more work. In particular, consider a connected component G' of G that remains after removing a T-cycle separator from G. In order to apply the result of Lipton and Tarjan to G', edges may need to be added to G' in such a way that it remains planar and such that L is still its breath first search layering. This is (at least in the case of planar graphs) possible and the explicit proof can be found in [5], where the layered separators in this form have been introduced. The authors used layered separators to show that planar graphs have non-repetitive chromatic number at most $\mathcal{O}(\log n)$ [5], thus breaking a long standing $\mathcal{O}(\sqrt{n})$ bound.

Lemma 1. *[5, 20] Let L be a breath first search layering of a triangulated (that is, edge maximal) planar graph G. Then G has a layered $(2, L)$-separator.*

Our main result is expressed in terms of track layouts of graphs, a type of graph layouts that is closely related to queue layouts and 3D grid drawings. We define track layouts first. A *vertex $|I|$-colouring* of a graph G is a partition $\{V_i : i \in I\}$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. The elements of the set I are *colours*, and each set V_i is a *colour class*. Suppose that $<_i$ is a total order on each colour class V_i. Then each pair $(V_i, <_i)$ is a *track*, and $\{(V_i, <_i) : i \in I\}$ is an $|I|$-*track assignment* of G. The *span* of an edge vw in a track assignment $\{(V_i, <_i) : 1 \leq i \leq t\}$ is $|i - j|$ where $v \in V_i$ and $w \in V_j$.

An *X-crossing* in a track assignment consists of two edges vw and xy such that $v <_i x$ and $y <_j w$, for distinct colours i and j. A t-track assignment of G that has no X-crossings is called t-*track layout* of G. The minimum t such that a graph G has t-track layout is called *track number* of G.

The main result of this paper is the following.

Theorem 1. *Every n-vertex graph G that has a layered ℓ-separator has track number at most $3\ell \lceil \log_{3/2} n \rceil + 3\ell$.*

All the other results mentioned earlier follow from this theorem and previously known results. Before proving Theorem 1 we discuss these implications.

If a graph G has a t-track layout with maximum edge span s, then the queue number of G is at most s and thus at most $t - 1$ [6]. Furthermore, every c-colourable t-track graph G with n vertices has a 3D grid drawing in $\mathcal{O}(t^2 n)$ volume [6] as well as in $\mathcal{O}(c^7 tn)$ volume [8]. Thus Theorem 1 implies.

Corollary 1. *Every n-vertex c-colourable graph G that has a layered ℓ-separator has queue number at most $3\ell\lceil\log_{3/2} n\rceil + 3\ell$ and a 3D grid drawing in $O(\log n)$ volume as well as in $\mathcal{O}(c^7 \ell\, n\log n)$ volume.*

Together with Lemma 1 this finally implies all the claimed results on planar graphs.

Corollary 2. *Every n-vertex planar graph has track number and queue number at most $6\lceil\log_{3/2} n\rceil + 6$ and a 3D grid drawing in $\mathcal{O}(n\log n)$ volume.*

3   Proof of Theorem 1

Let G be an n-vertex graph, let $L = \{V_0, V_1, \ldots, V_p\}$ be a layering of G, and let $\ell \geq 1$, such that G has (ℓ, L)–separator. Removing such a separator from G splits G into connected components each of which has at most $\frac{2}{3}|V(G)|$ vertices and its own (ℓ, L)–separator. Thus the process can continue until each component is an (ℓ, L)–separator of itself. This process naturally defines a rooted tree S and a mapping of $V(G)$ to the nodes of S, as follows. The root of S is a node to which the vertices of an (ℓ, L)–separator of G are mapped. The root has $c \geq 1$ children in S, one for each connected component G_j, $j \in [1, c]$, obtained by removing the (ℓ, L)–separator from G. The vertices of an (ℓ, L)–separator of G_j, $j \in [1, c]$, are mapped to a child of the root. The process continues until each component is an (ℓ, L)–separator of itself, or more specifically until each component has at most ℓ-vertices in each layer of L. In that case, such a component is an (ℓ, L)–separator of itself and its vertices are mapped to a leaf of S. This defines a rooted tree S and a partition of V to the nodes of S. One important observation, is that the height of S is most $\lceil\log_{3/2} n\rceil + 1$. For a node s of S, let $s(G)$ denote the set of vertices of G that are mapped to s and let $G[s]$ denote the graph induced by $s(G)$ in G. Note that for each node s of S, $s(G)$ has at most ℓ vertices in any layer of L.

Theorem 1 states that G has a track layout with $\mathcal{O}(\ell\log n)$ tracks. To prove this we will first create a track layout T of G with possibly lots of tracks. We then modify that layout in order to reduce the number of tracks to $\mathcal{O}(\ell\log n)$.

To ease the notation, for a track $(V_r, <_r)$, indexed by colour r, in a track assignment R, we denote that track by (r) when the ordering on each colour class is implicit. Also we sometimes write $v <_R w$. This indicates that v and w are on a same track r of R and that $v <_r w$.

Throughout this section, it is important to keep in mind that a *layer* is a subset of vertices of G defined by the layering L and that a *track* is an (ordered) subset of vertices of G defined by a track assignment of G.

We first define a track assignment T of G. Consult Figure 2 in the process. Each vertex v of $V(G)$ is assigned to a track whose colour is defined by three indices (d, i, k). Let s_v denote the node of the tree S that v is mapped to. The first index is the depth of s_v in S. The root is considered to have depth 1. Thus the first index, d, ranges from 1 to $\lceil\log_{3/2} n\rceil + 1$. The second index is the layer of L that contains v. Thus the second index, i, can be as big as $\Omega(n)$. Finally, $s_v(G)$ contains at most ℓ vertices from layer i in L. Label these, at most ℓ, vertices arbitrarily from 1 to ℓ and let the third index k of each of them be determined by this label. Consider the tracks themselves to be lexicographically ordered.
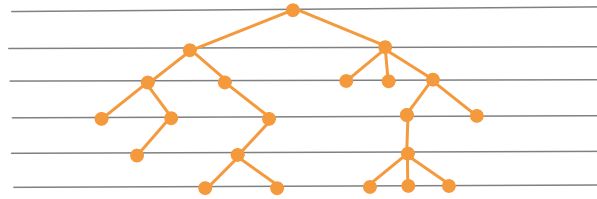
Figure 1: A track layout of the tree S.

To complete the track assignment we need to define the ordering of vertices in the same track. To do that we first define a simple track layout of the tree S. Consider a natural way to draw S in the plane without crossings such that all the nodes of S that are at the same distance from the root are drawn on the same horizontal line, as illustrated in Figure 1. This defines a track layout T_S of S where each horizontal line is a track and the ordering of the nodes within each track is implied by the crossing free drawing of S.

To complete the track assignment T, we need to define the total order of vertices that are in the same track of T. For any two vertices v and w of G that are assigned to the same track (d, i, k) in T, let $v <_T w$ if the node s_v that v maps to in S appears in T_S to the left of the node s_w that w maps to in S, that is, if $s_v <_{T_S} s_w$. Since v and w are in the same track of T only if they are mapped two distinct nodes of S that are the the same distance from the root of S, this defines a total order of each track in T. Figure 2 depicts the resulting track assignment T of G.

It is not difficult to verify that T is indeed a track layout of G, that is, T does not have X-crossings. This track layout however may have $\Omega(n)$ tracks. We now modify T to reduce the number of tracks to the claimed number.

For a vertex v of G, let (d_v, i_v, k_v) denote the track of v in T.

Dujmović and Wood [7], inspired by Felsner *et al.*[12], proved a simple wrapping lemma that says that a track layout with maximum span s can be wrapped into a $(2s + 1)$–track layout. For example, Figure 4 in the appendix, depicts how a track layout of a tree with maximum span 1 (like the track layout T_S of S) can be wrapped around a triangular prism to give a 3-track layout.

Unfortunately, the track layout T of G does not have a bounded span – its span can be $\Omega(n)$. (Since the tracks of T are ordered by lexicographical ordering, span is well defined in T.) However parts of the layout do have bounded span. In particular, consider the graph, G_d induced by the vertices of G that are assigned to the tracks of T that have the same first index, d. For each d, the tracks of T with that first index equal to d, define a track layout T_d of G_d, as illustrated, for $d = 2$ case, in the top part of Figure 3. Recall that G_d is comprised of disjoint layered (l, L)-separators (see Figure 2). Since each edge in a layered (l, L)-separator either connects two vertices in the same layer of L or two vertices from two consecutive layers of L, the span of an edge of G_d in T_d is at most $2\ell - 1$.
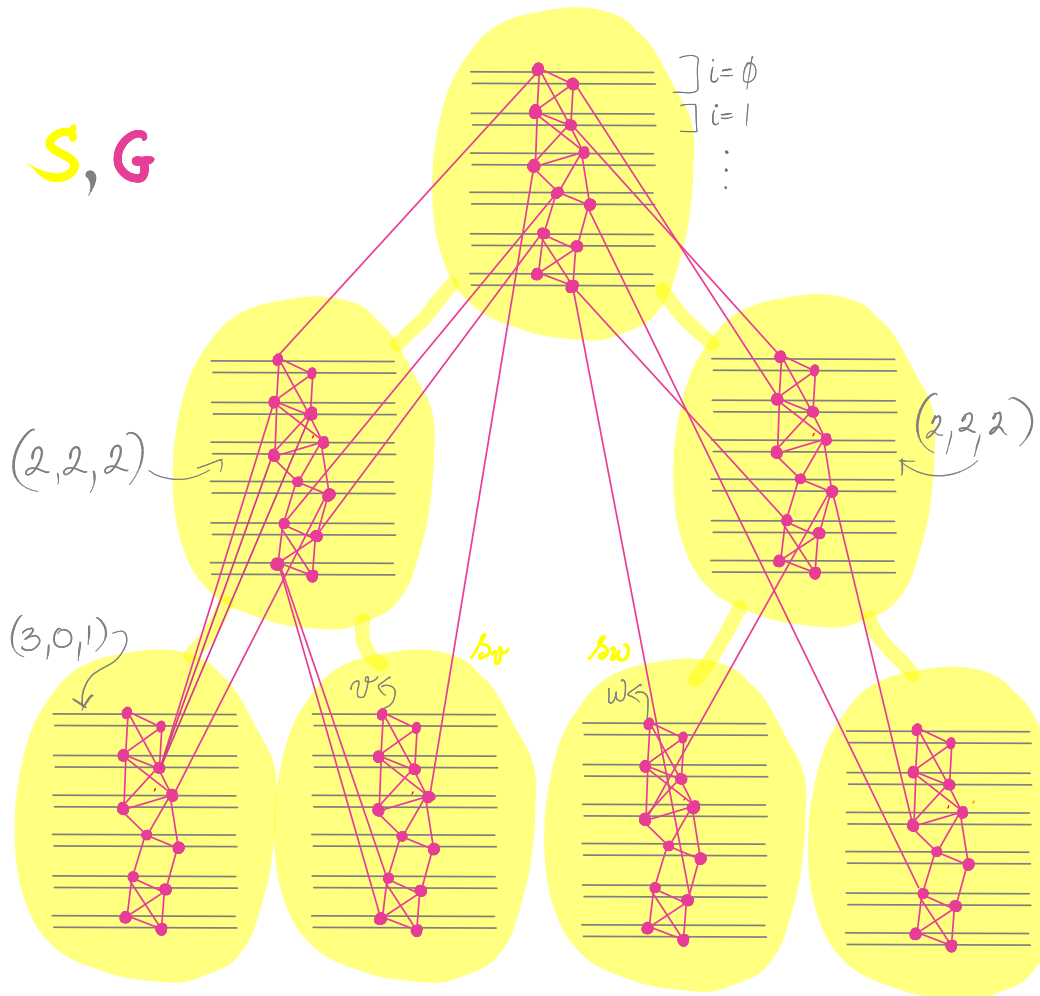
S, G

$i = \emptyset$
$i = 1$
$\vdots$

$(2,2,2)$

$(2,2,2)$

$(3,0,1)$

b_v b_w

$v \subset$ $w \subset$

Figure 2: A track layout T of a graph G which has a layered ($\ell = 2$)–separator.
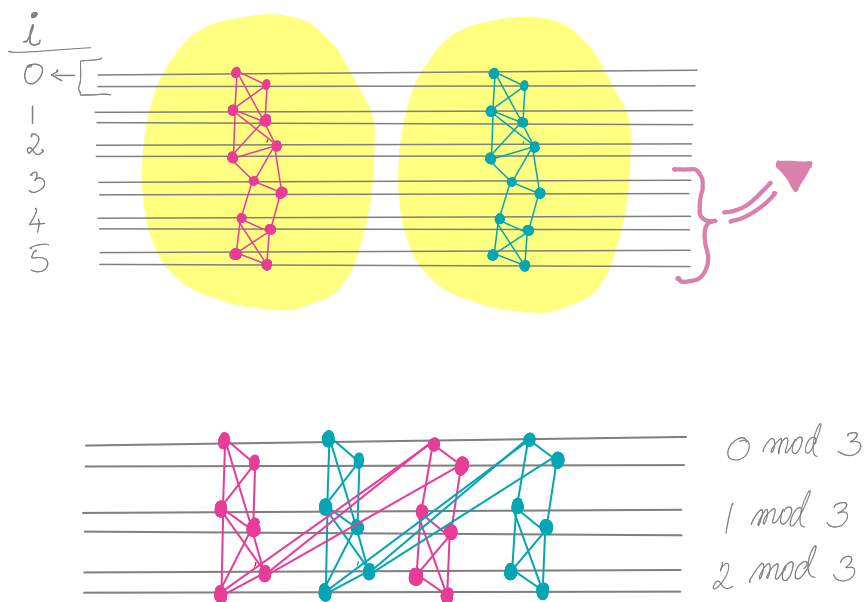

i
0
1
2
3
4
5

0 mod 3
1 mod 3
2 mod 3

Figure 3: Top figure: the track layout T_2 of G_2. Bottom figure: the track layout T_2' obtained by wrapping T_2.

For each d, we now wrap the track layout T_d into a 3ℓ-track layout T'_d of G_d, as illustrated in the bottom of Figure 3. The exact version of the wrapping lemma we use is given in the appendix, see Lemma 2. It mimics the wrapping of Felsner *et al.*[12] and it is included for completeness. This defines a track assignment T' of G.

(One may be tempted to, instead of wrapping all of G_d, wrap individually the track layouts of the graphs induced by the vertices mapped to the same node of S. It can be shown however that such strategy can introduce X-crossings in T'.)

The wrapping lemma, Lemma 2, implies that for all d, T'_d has the following useful properties. Consider two vertices a and b that are in the same track $f' = (d_a, i_a \mod 3, k_a) = (d_b, i_b \mod 3, k_b)$ in T'_d. Then if $a <_{f'} b$ in T'_d and

(1) $i_a \neq i_b$ then $i_b \geq i_a + 3$,

(2) otherwise, ($i_a = i_b$), a and b were in the same track f in T and $a <_f b$. (This is because the wrapping does not change the ordering of vertices that were already in the same track in T).

Since $d \leq \lceil \log_{3/2} n \rceil + 1$, $i \mod 3 \leq 3$ and $k \leq \ell$, the track assignment T' of G has at most $3\ell \lceil \log_{3/2} n \rceil + 1$ tracks, as claimed. It remains to prove that T' is in fact a track layout of G, that is, there are no X-crossings in the track assignment T'

Assume by contradiction that there are two edges vw and xy that form an X-crossing in T'. Let v and x belong to a same track in T' and let y and w belong to a same track in T'. If $d_v = d_w = d_x = d_y$, then v, w, x and y belong to the the same graph G_d and thus they do not form and X-crossing since T'_d does not have X-crossings by the wrapping lemma.

Thus $d_v = d_x = d_1$ and $d_w = d_y = d_2$ and $d_1 \neq d_2$. Let without loss of generality $d_1 < d_2$ and $v <_{T'} x$ and $y <_{T'} w$ in T'. Since w and y are in the same track, $d_y = d_w$, $k_y = k_w$ and either $i_y = i_w$ or $i_w \geq i_y + 3$ by properties (1) and (2). There are thus two cases to consider. First consider the case that $i_w \geq i_y + 3$. Since w is adjacent to v, $i_v = \{i_w - 1, i_w, i_w + 1\}$ and similarly $i_x = \{i_y - 1, i_y, i_y + 1\}$. Thus $i_v \geq i_w - 1 \geq i_y + 2$ and $i_x \leq i_y + 1$. Thus, $i_v > i_x$ and property (1) applies to v and x. This contradicts the assumption that $v <_{T'} x$, since property (1), implies that $i_x > i_v$.

Finally, consider the case that $i_y = i_w$. Then y and w are in the same track in T and their ordering, $y <_{T'} w$, in T' is the same as in T, $y <_T w$, by property (2). Since v and x are in the same track in T' and $v <_{T'} x$, either $i_v = i_x$ or $i_x \geq i_v + 3$, by properties (1) and (2). Thus again, since w is adjacent to v, $i_v = \{i_w - 1, i_w, i_w + 1\}$ and similarly $i_x = \{i_y - 1, i_y, i_y + 1\}$. Since $i_y = i_w$, no pair of these indices differs by at least 3 and thus $i_v = i_x$ by property (1). That implies that v and x are in the same track in T and thus by property (2) their ordering in T', $v <_{T'} x$, is the same as in T, $v <_T x$. This implies that vw and xy form an X-crossing in T thus providing the desired contradiction. This completes the proof of Theorem 1. $\square$

4 Conclusions

Classical separators are one of the most powerful and widely used tools in graph theory as they allow for solving all kinds of combinatorial problems with the divide and conquer

method. However, for most of the interesting graph classes such separators can be polynomial in n, such as $\Omega(\sqrt{n})$ for planar graphs, thus it is unclear how they can help in proving sub-polynomial bounds.

Layered separators provide extra structure that can aid in attacking problems that inherently look for some vertex ordering, such as in queue layouts and to lesser extent in non-repetitive colourings – and this despite of the fact that such separators can be of linear order. The critical feature of layered separators is that edges only appear between consecutive layers and that the number of vertices per layer is bounded. As we have seen, planar graphs have such separators. It is an interesting question to determine what other classes of graphs have layered $\mathcal{O}(1)$-separators, especially given the implications such separators have on the aforementioned problems. The graphs of bounded genus and more generally all proper minor closed families are the natural candidates.

Related to that, is the result of Gilbert, Hutchinson and Tarjan (see the proof of Theorem 4 [13]) who showed that every graph G of genus g has $2g$ T-cycles whose removal from G leaves a planar graph G'. However, (recall the discussion from Section 2) it is unclear if G' can be made edge-maximal planar while keeping L as its breath first search layering.

Finally, the way the layered $\mathcal{O}(1)$-separators are used in this paper, and in [5], inherently leads to logarithmic upper bounds. It seems difficult to envision how such separators could be used to obtain sub-logarithmic upper bounds. Thus, if these $\mathcal{O}(\log n)$ upper bounds are not tight, the new methods may be needed to break them.

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A   Wrapping Lemma

For completeness, we add the version of the wrapping lemma used in this paper. The original lemma of Felsner *et al.*[11, 12] is the below lemma with $\ell = 1$. Figure 4 below also depicts $\ell = 1$ case.

Lemma 2. *[7, 12] Let T denote a track layout of a graph G with tracks in T indexed by (i, k) where $i \in \{0, \ldots, p\}$ and $k \in \{1, \ldots, \ell\}$ and such that for each edge vw of G, with v in track (i_v, k_v) and w in (i_w, k_w), $|i_v - i_w| \leq 1$. Then T can be modified (wrapped) into a 3ℓ track layout T' of G as follows: Each vertex v of G is assigned to a track $(i_v \mod 3, k_v)$ and two vertices v and x that are in the same track of T' are ordered as follows. Let $i_v \leq i_x$.*
(1) If $i_v < i_x$, then $v <_{T'} x$.
(2) Otherwise, $(i_v = i_x)$, v and x are ordered in T' as in T.

Proof. It is simple to verify that each track in the track assignment T' is a total order. It remains to prove that T' has no X-crossings. Assume by contradiction that vw and xy form an X-crossing in T'. Without loss of generality let $v <_{T'} x$ and $y <_{T'} w$. Since $i_v \mod 3 = i_x \mod 3$, either $i_v = i_x$, or $i_x \geq i_v + 3$.

If $i_v = i_x$, then $v <_T x$. Since v is adjacent to w, $i_w = \{i_v - 1, i_v, i_v + 1\}$ and similarly $i_y = \{i_x - 1, i_x, i_x + 1\}$. No pair of these indices differs by at least 3 when $i_v = i_x$, thus $|i_w - i_y| \leq 2$. Since w and y are in the same track in T', $i_w \mod 3 = i_y \mod 3$. Together with $|i_w - i_y| \leq 2$, this implies that $i_w = i_y$. That implies further that y and w are in the same track in T and thus by property (2) their ordering $y <_{T'} w$ in T', is the same as in T. Thus $y <_T w$, which implies that vw and xy form an X-crossing in T thus contradicting the assumption.

If $i_v < i_x$, then $i_x \geq i_v + 3$. Again, since v is adjacent to w, $i_w = \{i_v - 1, i_v, i_v + 1\}$ and similarly $i_y = \{i_x - 1, i_x, i_x + 1\}$. Thus $i_w \leq i_v + 1$ and $i_y \geq i_v + 2$. Therefore, $i_w < i_y$ and by property (1), $w <_{T'} y$, contradicting the above assumption on the ordering of y and w in T'. $\square$
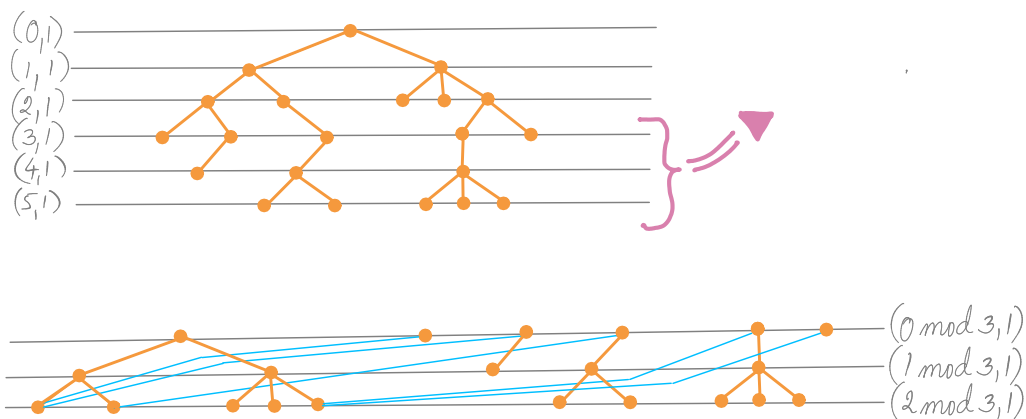

Figure 4: Top figure: a 6-track span-1 layout of a tree. Bottom figure: after wrapping to a 3-track layout.