# Really Straight Graph Drawings*

Vida Dujmović [†,‡] Matthew Suderman [†] David R. Wood [‡,§]

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**Abstract**

We study straight-line drawings of graphs with few segments and few slopes. Optimal results are obtained for all trees. Tight bounds are obtained for outerplanar graphs, 2-trees, and planar 3-trees. We prove that every 3-connected plane graph on $n$ vertices has a plane drawing with at most $5n/2$ segments and at most $2n$ slopes. We prove that every cubic 3-connected plane graph has a plane drawing with three slopes (and three bends on the outerface). Drawings of non-planar graphs with few slopes are also considered. For example, interval graphs, co-comparability graphs and AT-free graphs are shown to have have drawings in which the number of slopes is bounded by the maximum degree. We prove that graphs of bounded degree and bounded treewidth have drawings with $\mathcal{O}(\log n)$ slopes. Finally we prove that every graph has a drawing with one bend per edge, in which the number of slopes is at most one more than the maximum degree.

[†]School of Computer Science, McGill University, Montréal, Canada ({vida,suderman}@cs.mcgill.ca).

[‡]School of Computer Science, Carleton University, Ottawa, Canada (davidw@scs.carleton.ca).

[§]Department of Applied Mathematics, Charles University, Prague, Czech Republic.

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# Contents

# 1 Introduction

A common requirement for an aesthetically pleasing drawing of graph is that the edges are straight. This paper studies the following additional requirements of straight-line graph drawings:

1. minimise the number of segments in the drawing

2. minimise the number of distinct edge slopes in the drawing

First we formalise these notions. Consider a mapping of the vertices of a graph to distinct points in the plane. Now represent each edge by the closed line segment between its endpoints. Such a mapping is a (*straight-line*) *drawing* if each edge does not intersect any vertex, except for its own endpoints. By a *segment* in a drawing, we mean a maximal set of edges that form a line segment. The *slope* of a line $L$ is the angle swept from the X-axis in an anticlockwise direction to $L$ (and is thus in $[0, \pi)$). The *slope* of an edge or segment is the slope of the line that extends it. Of course two edges have the same *slope* if and only if they are parallel. A *crossing* in a drawing is a pair of edges that intersect at some point other than a common endpoint. A drawing is *plane* if it has no crossings. A *plane graph* is a planar graph with a fixed combinatorial embedding and a specified outerface. We emphasise that a plane drawing of a plane graph must preserve the embedding and outerface. That every plane graph has a plane drawing is a famous result independently due to Wagner [53] and Fáry [21].

In this paper we prove lower and upper bounds on the minimum number of segments and slopes in (plane) drawings of graphs. A summary of our results is given in Table 1. A number of comments are in order when considering these results: (1) The minimum number of slopes in a drawing of (plane) graph $G$ is at most the minimum number of segments in a drawing of $G$. (2) Upper bounds for plane graphs are stronger than for planar graphs, since for planar graphs one has the freedom to choose the embedding and outerface. On the other hand, lower bounds for planar graphs are stronger than for plane graphs. (3) Deleting an edge in a drawing cannot increase the number of slopes, whereas it can increase the number of segments. Thus, the upper bounds for slopes are applicable to all subgraphs of the mentioned graph families, unlike the upper bounds for segments.

The paper is organised as follows. In Section 2 we prove some elementary lower bounds on the number of segments and slopes in a drawing. We also show that it is $\mathcal{NP}$-complete to determine whether a graph has a plane drawing on two slopes.

Section 3 studies plane drawings of graphs with small treewidth. In particular, we consider trees, outerplanar graphs, 2-trees, and planar 3-trees. For any tree, we construct a plane drawing with the minimum number of segments and the minimum number of slopes. For outerplanar graphs, 2-trees, and planar 3-trees, we determine bounds on the minimum number of segments and slopes that are tight in the worst-case.

Section 4 studies plane drawings of 3-connected plane and planar graphs. In the case of slope-minimisation for plane graphs we obtain a bound that is tight in the worst case. However, our lower bound examples have linear maximum degree. We drastically improve the upper bound in the case of cubic graphs. We prove that every 3-connected plane cubic graph has a plane drawing with three slopes, except for three edges on the outerface that

Table 1: Summary of results (ignoring additive constants). Here $n$ is the number of vertices, $\eta$ is the number of vertices of odd degree, and $\delta$ and $\Delta$ are the minimum and maximum degree. In the upper half of the table, the drawings are plane. The lower bounds are existential, except for trees, for which the lower bounds are universal.

| graph family | # segments | | # slopes | |
|---|---|---|---|---|
| | $\geq$ | $\leq$ | $\geq$ | $\leq$ |
| trees | $\eta/2$ | $\eta/2$ | $\lceil\Delta/2\rceil$ | $\lceil\Delta/2\rceil$ |
| maximal outerplanar | $n$ | $n$ | - | $n$ |
| plane 2-trees | $2n$ | $2n$ | $2n$ | $2n$ |
| plane 3-trees | $2n$ | $2n$ | $2n$ | $2n$ |
| plane 2-connected | $5n/2$ | - | $2n$ | - |
| planar 2-connected | $2n$ | - | $n$ | - |
| plane 3-connected | $2n$ | $5n/2$ | $2n$ | $2n$ |
| planar 3-connected | $2n$ | $5n/2$ | $n$ | $2n$ |
| plane 3-connected cubic | - | $n+2$ | 3 | 3 |
| all graphs | $\eta/2$ | - | $\max\{\delta, \lceil\Delta/2\rceil\}$ | - |
| complete graph | $\binom{n}{2}$ | | $n$ | $n$ |
| balanced complete bipartite graph | $\Theta(n^2)$ | | $n/2$ | $n/2$ |
| bandwidth $b$ | - | - | $\Omega(b)$ | $\mathcal{O}(b^2)$ |
| interval graphs | - | - | $\lceil\Delta/2\rceil$ | $\mathcal{O}(\Delta^2)$ |
| co-comparability graphs | - | - | $\lceil\Delta/2\rceil$ | $\mathcal{O}(\Delta^2)$ |
| AT-free graphs | - | - | $\lceil\Delta/2\rceil$ | $\mathcal{O}(\Delta^2)$ |
| bounded degree, bounded treewidth | - | - | - | $\mathcal{O}(\log n)$ |

have their own slope. As a corollary we prove that every 3-connected plane cubic graph has a plane 'drawing' with three slopes and three bends on the outerface.

The remainder of the paper studies non-plane drawings with few slopes. Section 5 deals with complete multipartite graphs, and Section 6 considers drawings of arbitrary graphs. For example, we prove that every graph with bounded degree and bounded treewidth has a drawing with $\mathcal{O}(\log n)$ slopes. In Section 7 we consider 1-bend drawings with few slopes.

### 1.1 Related Research

We now outline some related research from the literature.

- As illustrated in Figure 1, Eppstein [18] characterised those planar graphs that have plane drawings with a segment between every pair of vertices. In some sense, these are the plane drawings with the least number of slopes.

- Drawings of lattices and posets with few slopes have been considered by Czyzowicz *et al.* [6, 7, 8] and Freese [23].

- A famous result by Ungar [51], settling an open problem of Scott [46], states that

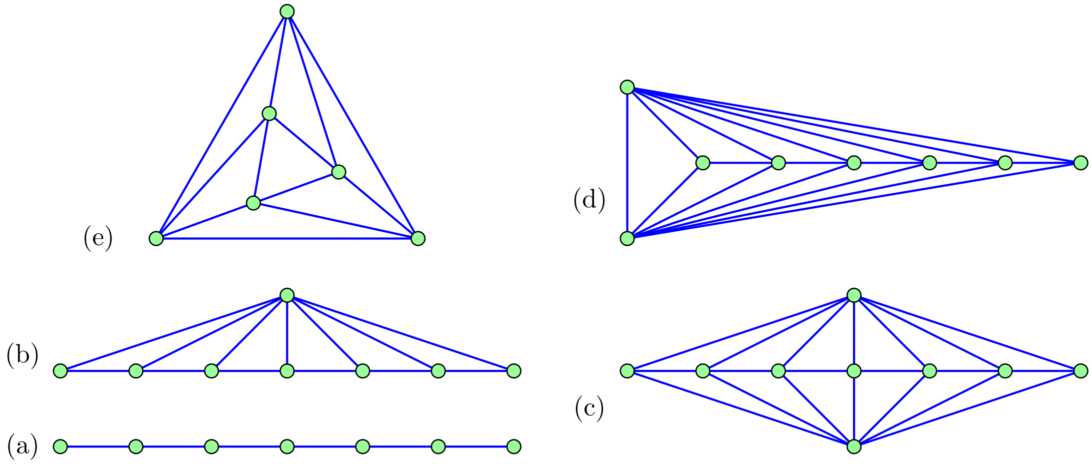


Figure 1: The plane graphs with a segment between every pair of vertices.

$n$ non-collinear points determine at least $n-1$ distinct slopes. The configurations of $n$ points that determine exactly $n-1$ distinct slopes have been investigated by Jamison [27, 28]. Jamison [30] generalised the result of Ungar by proving that any set of non-collinear points has a spanning tree whose edges have distinct slopes.

- Wade and Chu [52] recognised that drawing arbitrary graphs with few slopes is an interesting problem. They defined the *slope-number* of a graph $G$ to be the minimum number of slopes in a drawing of $G$. However, the results of Wade and Chu [52] were only for complete graphs, and these results were well known in the discrete geometry literature (see Section 5).

- The *geometric thickness* of a graph $G$ is the minimum $k$ such that $G$ has a drawing in which every edge receives one of $k$ colours, and monochromatic edges do not cross (see [10, 16, 19]). That is, each colour class is a plane subgraph. In any drawing, edges with the same slope do not cross. Thus the geometric thickness of $G$ is a lower bound on the minimum number of slopes in a drawing of $G$.

- A drawing is *convex* if all the vertices are on the convex hull, and no three vertices are collinear. The *book thickness* of a graph (also called *pagenumber* and *stacknumber*) is the same as geometric thickness except that the drawing must be convex (see [15] for numerous references). Since edges with the same slope do not cross, the book thickness of $G$ is a lower bound on the minimum number of slopes in a convex drawing of $G$.

- Plane orthogonal drawings with two slopes (and few bends) have been extensively studied [1, 2, 40, 41, 42, 43, 44, 49, 50]. For example, Ungar [50] proved that every cyclically 4-edge-connected plane cubic graph has a plane drawing with two slopes and four bends on the outerface. Thus our result for 3-connected plane cubic graphs (Corollary 3) nicely complements this theorem of Ungar.

- The maximum number of vertices in a grid that determine a set of slopes, no two of which are the same, has been studied by Erdős *et al.* [20], Peile and Taylor [39], and Zhang [55].

- Martin [33, 34, 35] has theoretical results on graph drawings related to slope.

- Multi-dimensional graph drawings with few segments or few slopes is also of interest. Since an orthogonal projection preserves parallel lines, and since there always is a 'nice' orthogonal projection from $d \geq 3$ dimensions into the plane, the best bounds on the number of slopes and segments are obtained in two dimensions. Here a projection is 'nice', if no vertex-vertex or vertex-edge occlusions occur; see [5, 17, 26]. Thus multi-dimensional drawings with few segments or few slopes are only interesting if the vertices are restricted to not all lie in a single plane. Under thus assumption, Pach *et al.* [37, 38] recently proved that the minimum number of slopes in a three-dimensional drawing of the complete graph $K_n$ is $2n-3$. The proof is based on a generalisation of the above-mentioned result of Ungar [51]. In related work, Onn and Pinchasi [36] studied the minimum number of edge-slopes in a $d$-dimensional convex polytope.

### 1.2 Definitions

We consider undirected, finite, and simple graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of $G$ are respectively denoted by $n = |V(G)|$ and $m = |E(G)|$. The maximum degree of $G$ is denoted by $\Delta(G)$.

For all $S \subseteq V(G)$, the (*vertex-*) *induced* subgraph $G[S]$ has vertex set $S$ and edge set $\{vw \in E(G) : v, w \in S\}$. For all $S \subseteq V(G)$, let $G \setminus S$ be the subgraph $G[V(G) \setminus S]$. For all $v \in V(G)$, let $G \setminus v = G \setminus \{v\}$. For all $A, B \subseteq V(G)$, let $G[A, B]$ be the bipartite subgraph of $G$ with vertex set $A \cup B$ and edge set $\{vw \in E(G) : v \in A \setminus B, w \in B \setminus A\}$.

For all $S \subseteq E(G)$, the (*edge-*) *induced* subgraph $G[S]$ has vertex set $\{v \in V(G) : \exists\, vw \in S\}$ and edge set $S$. For all pairs of vertices $v, w \in V(G)$, let $G \cup vw$ be the graph with vertex set $V(G)$ and edge set $E(G) \cup \{vw\}$.

Let $G$ be a graph and let $T$ be a tree. An element of $V(T)$ is called a *node*. Let $\{T_x \subseteq V(G) : x \in V(T)\}$ be a set of subsets of $V(G)$ indexed by the nodes of $T$. Each $T_x$ is called a *bag*. The pair $(T, \{T_x : x \in V(T)\})$ is a *tree decomposition* of $G$ if:

- $\bigcup_{x \in V(T)} T_x = V(G)$ (that is, every vertex of $G$ is in at least one bag),

- $\forall$ edge $vw$ of $G$, $\exists$ node $x$ of $T$ such that $v \in T_x$ and $w \in T_x$, and

- $\forall$ nodes $x, y, z$ of $T$, if $y$ is on the path from $x$ to $z$ in $T$, then $T_x \cap T_z \subseteq T_y$.

The *width* of a tree decomposition is one less than the maximum cardinality of a bag. A *path decomposition* is a tree-decomposition where the tree $T$ is a path $T = (x_1, x_2, \ldots, x_m)$, which is simply identified by the sequence of bags $T_1, T_2, \ldots, T_m$ where each $T_i = T_{x_i}$. The *pathwidth* (respectively, *treewidth*) of a graph $G$ is the minimum width of a path (tree) decomposition of $G$.

For each integer $k \geq 1$, $k$-trees are the class of graphs defined recursively as follows. The complete graph $K_{k+1}$ is a *$k$-tree*, and the graph obtained from a $k$-tree by adding a new vertex adjacent to each vertex of an existing $k$-clique is also a *$k$-tree*. It is well known that the treewidth of a graph $G$ equals the minimum $k$ such that $G$ is the spanning subgraph of a $k$-tree.

For example, the graphs of treewidth one are the forests. Graphs of treewidth two, called *series-parallel*, are planar since in the construction of a 2-tree, each new vertex can be drawn close to the midpoint of the edge that it is added onto. Maximal outerplanar graphs are examples of 2-trees.

## 2 Lower Bounds

The following result is immediate, as illustrated in Figure 2.

**Lemma 1.** *Let $u$, $v$ and $w$ be three non-collinear vertices in a drawing $D$ of a graph $G$. Let $d(u)$ denote the number of edges incident to $u$ that intersect the interior of the triangle $uvw$, and similarly for $v$ and $w$. Then $D$ has at least $d(u)+d(v)+d(w)+|E(G)\cap\{uv,vw,uw\}|$ slopes.* □

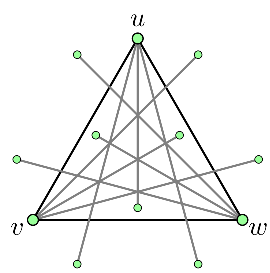


Figure 2: A triangle forces many different slopes.

We have the following lower bounds.

**Lemma 2.** *The number of segments in a drawing of a graph is at least half the number of odd degree vertices.*

*Proof.* If a vertex is internal on every segment then it has even degree. Thus each vertex of odd degree is an endpoint of some segment. Thus the number of vertices of odd degree is at most twice the number of segments. (The number of odd degree vertices is always even.) □

**Lemma 3.** *The number of slopes in a drawing of a graph is (a) at least half the maximum degree, and (b) at least the minimum degree. The number of slopes in a convex drawing of a graph is (c) at least the maximum degree.*

*Proof.* At most two edges incident to a single vertex can have the same slope. This proves (a). In a drawing $D$ of a (finite) graph, there is a line $L$ such that $D\cap L$ consists of exactly one vertex $v$. Thus all of the edges incident to $v$ have distinct slopes. This proves (b). Every edge incident to a single vertex in a convex drawing has distinct slope. This proves (c). □

## 2.1 Computational Complexity

**Lemma 4.** *A graph has a (plane) drawing on two slopes if and only if it has a (plane) drawing on any two slopes.*

*Proof.* It suffices to prove that a (plane) drawing $D_a$ of $G$ with slopes 0 and $\alpha$ can be converted into a (plane) drawing $D_b$ of $G$ with slopes 0 and $\beta$. Create a grid $P_{0,\alpha}$ on $D_a$ as follows. At each vertex $v \in V(G)$ draw two lines, one with slope 0 and one with slope $\alpha$. Each vertex $v$ has some coordinate $P_{0,\alpha}(i,j)$ and each edge follows straight gridlines. Now create a grid $P_{0,\beta}$ with slopes 0 and $\beta$. Place each vertex that was at $P_{0,\alpha}(i,j)$ at $P_{0,\beta}(i,j)$. Clearly, in $P_{0,\beta}$ all the edges follow straight gridlines. Furthermore, no edge contains a vertex in its interior and two edges cross in $P_{0,\alpha}(i,j)$ if and only if they cross in $P_{0,\beta}(i,j)$. Thus we obtain a (plane) drawing $D_b$ of $G$ with slopes 0 and $\beta$. □

Garg and Tamassia [24] proved that it is $\mathcal{NP}$-complete to decide whether a graph has a rectilinear planar drawing (that is, with vertical and horizontal edges). Thus Lemma 4 implies:

**Corollary 1.** *It is $\mathcal{NP}$-complete to decide whether a graph has a plane drawing with two slopes.* □

# 3 Planar Graphs with Small Treewidth

## 3.1 Trees

**Theorem 1.** *Let $T$ be a tree with maximum degree $\Delta$, and with $\eta$ vertices of odd degree. The minimum number of segments in a drawing of $T$ is $\eta/2$. The minimum number of slopes in a drawing of $T$ is $\lceil \Delta/2 \rceil$. Moreover, $T$ has a plane drawing with $\eta/2$ segments and $\lceil \Delta/2 \rceil$ slopes.*

*Proof.* The lower bounds are from Lemmata 2 and 3(a). The upper bound will follow from the following hypothesis, which we prove by induction on the number of vertices: "Every tree $T$ with maximum degree $\Delta$ has a plane drawing with $\lceil \Delta/2 \rceil$ slopes, in which every odd degree vertex is an endpoint of exactly one segment, and no even degree vertex is an endpoint of a segment." The hypothesis is trivially true for a single vertex. Let $x$ be a leaf of $T$ incident to the edge $xy$. Let $T' = T \setminus x$. Suppose $T'$ has maximum degree $\Delta'$.

First suppose that $y$ has even degree in $T$, as illustrated in Figure 3(a). Thus $y$ has odd degree in $T'$. By induction, $T'$ has a plane drawing with $\lceil \Delta'/2 \rceil \leq \lceil \Delta/2 \rceil$ slopes, in which $y$ is an endpoint of exactly one segment. That segment contains some edge $e$ incident to $y$. Draw $x$ on the extension of $e$ so that there are no crossings. In the obtained drawing $D$, the number of slopes is unchanged, $x$ is an endpoint of one segment, and $y$ is not an endpoint of any segment. Thus $D$ satisfies the hypothesis.

Now suppose that $y$ has odd degree in $T$, as illustrated in Figure 3(b). Thus $y$ has even degree in $T'$. By induction, $T'$ has a plane drawing with $\lceil \Delta'/2 \rceil$ slopes, in which $y$ is not an endpoint of any segment. Thus the edges incident to $y$ use $\deg_{T'}(y)/2 \leq \lceil \Delta/2 \rceil - 1$ slopes. If the drawing of $T'$ has any other slopes, let $s$ be one of these slopes, otherwise let $s$ be an unused slope. Add edge $xy$ to the drawing of $T'$ with slope $s$ so that there are

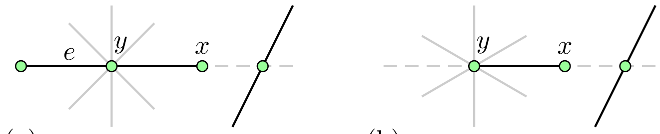


Figure 3: Adding a leaf $x$ to a drawing of a tree: (a) $\deg(y)$ even and (b) $\deg(y)$ odd.

no crossings. In the obtained drawing $D$, there is a new segment with endpoints $x$ and $y$. Since both $x$ and $y$ have odd degree in $T$, and since $x$ and $y$ were not endpoints of any segment in the drawing of $T'$, the induction hypothesis is maintained. The number of slopes in $D$ is $\max\{\lceil\Delta'/2\rceil, \deg_{T'}(y)/2+1\} \leq \lceil\Delta/2\rceil$. $\square$

### 3.2 Outerplanar Graphs

A planar graph $G$ is *outerplanar* if $G$ admits a combinatorial embedding with all the vertices on the boundary of a single face. An outerplanar graph $G$ is *maximal* if $G \cup vw$ is not outerplanar for any pair of non-adjacent vertices $v, w \in V(G)$. A plane graph is *outerplanar* if all the vertices are on the boundary of the outerface. A maximal outerplanar graph has a unique outerplanar embedding.

**Theorem 2.** *Every $n$-vertex maximal outerplanar graph $G$ has an outerplanar drawing with at most $n$ segments. For all $n \geq 3$, there is an $n$-vertex maximal outerplanar graph that has at least $n$ segments in any drawing.*

*Proof.* We prove the upper bound by induction on $n$ with the additional invariant that the drawing is *star-shaped*. That is, there is a point $p$ in (the interior of) some internal face of $D$, and every ray from $p$ intersects the boundary of the outerface in exactly one point.

For $n = 3$, $G$ is a triangle, and the invariant holds by taking $p$ to be any point in the internal face. Now suppose $n > 3$. It is well known that $G$ has a degree-2 vertex $v$ whose neighbours $x$ and $y$ are adjacent, and $G' = G \setminus v$ is maximal outerplanar. By induction, $G'$ has a drawing $D'$ with at most $n-1$ segments, and there is a point $p$ in some internal face of $D'$, such that every ray from $p$ intersects the boundary of $D'$ in exactly one point. The edge $xy$ lies on the boundary of the outerface and of some internal face $F$. Without loss of generality, $xy$ is horizontal in $D'$, and $F$ is below $xy$. Since $G'$ is maximal outerplanar, $F$ is bounded by a triangle $rxy$.

For three non-collinear points $a$, $b$ and $c$ in the plane, define the *wedge* $(a, b, c)$ to be the infinite region that contains the interior of the triangle $abc$, and is enclosed on two sides by the ray from $b$ through $a$ and the ray from $b$ through $c$. By induction, $p$ is in the wedge $(y, x, r)$ or in the wedge $(x, y, r)$. By symmetry we can assume that $p$ is in $(y, x, r)$.

Let $R$ be the region strictly above $xy$ that is contained in the wedge $(x, p, y)$. The line extending the edge $xr$ intersects $R$. As illustrated in Figure 4, place $v$ on any point in $R$ that is on the line extending $xr$. Draw the two incident edges $vx$ and $vy$ straight. This defines our drawing $D$ of $G$. By induction, $R \cap D' = \emptyset$. Thus $vx$ and $vy$ do not create crossings in $D$. Every ray from $p$ that intersects $R$, intersects the boundary of $D$ in exactly one point. All other rays from $p$ intersect the same part of the boundary of

$D$ as in $D'$. Since $p$ remains in some internal face, $D$ is star-shaped. By induction, $D'$ has $n-1$ segments. Since $vx$ and $rx$ are in the same segment, there is at most at one segment in $D \setminus D'$. Thus $D$ is a star-shaped outerplanar drawing of $G$ with $n$ segments. This concludes the proof of the upper bound.

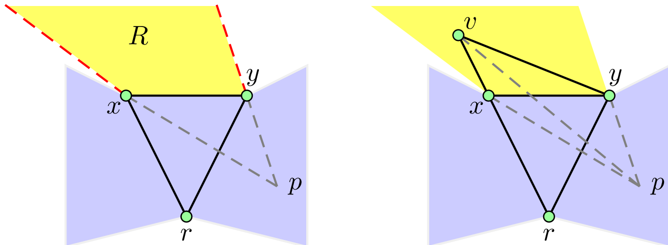


Figure 4: Construction of a star-shaped drawing of an outerplanar graph.

For the lower bound, let $G_n$ be the maximal outerplanar graph on $n \geq 3$ vertices whose weak dual (that is, dual graph disregarding the outerface) is a path and the maximum degree of $G_n$ is at most four, as illustrated in Figure 5.

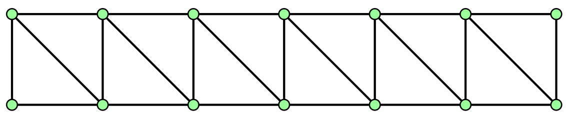

Figure 5: The graph $G_{14}$.

We claim that every drawing of $G_n$ has at least $n$ segments (even if crossings are allowed). We proceed by induction on $n$. The result is trivial for $n = 3$. Suppose that every drawing of $G_{n-1}$ has at least $n-1$ segments, but there exists a drawing $D$ of $G_n$ with at most $n-1$ segments. Let $v$ be a degree-2 vertex in $G_n$ adjacent to $x$ and $y$. One of $x$ and $y$, say $x$, has degree three in $G_n$. Observe that $G_n \setminus v$ is isomorphic to $G_{n-1}$. Thus we have a drawing of $G_n$ with exactly $n-1$ segments, which contains a drawing of $G_n \setminus v$ with $n-1$ segments. Thus the edge $vx$ shares a segment with some other edge $xr$, and the edge $vy$ shares a segment with some other edge $ys$. Since $vxy$ is a triangle, $r \neq y$, $s \neq x$ and $r \neq s$. Since $x$ has degree three, $y$ is adjacent to $r$, as illustrated in Figure 6. That accounts for all edges incident to $y$ and $x$. Thus $xy$ is a segment in $D$.

Now construct a drawing $D'$ of $G_{n-1}$ with $x$ moved to the position of $v$ in the drawing of $G_n$. The drawing $D$ consists of $D'$ plus the edge $xy$. Since $xy$ is a segment in $D$, $D'$ has one less segment than $D$. Thus $D'$ is a drawing of $G_{n-1}$ with at most $n-2$ segments, which is the desired contradiction. □

**Open Problem 1.** Is there a polynomial time algorithm to compute an outerplanar drawing of a given outerplanar graph with the minimum number of segments?

### 3.3 2-Trees and Planar 3-Trees

**Lemma 5.** *Every $n$-vertex 2-tree has a plane drawing with at most $2n-3$ segments. For all $n \geq 3$, there is an $n$-vertex plane 2-tree that has at least $2n-3$ slopes (and thus at least $2n-3$ segments) in every plane drawing.*

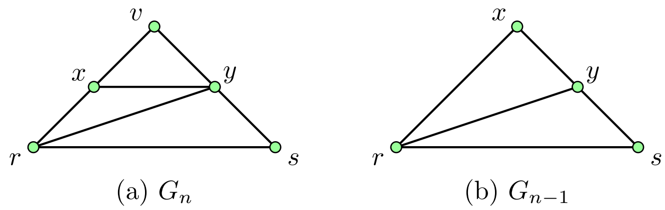


Figure 6: Construction of a drawing of $G_{n-1}$ from a drawing of $G_n$.

*Proof.* The upper bound follows from the Fáry/Wagner theorem since every 2-tree is planar and has $2n-3$ edges. Consider the 2-tree $G_n$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1v_2, v_1v_i, v_2v_i : 3 \leq i \leq n\}$. Fix a plane embedding of $G_n$ with the edge $v_1v_2$ on the triangular outerface, as illustrated in Figure 7(a). The number of slopes is at least $(n-3)+(n-3)+0+3 = 2n-3$ by Lemma 1. □

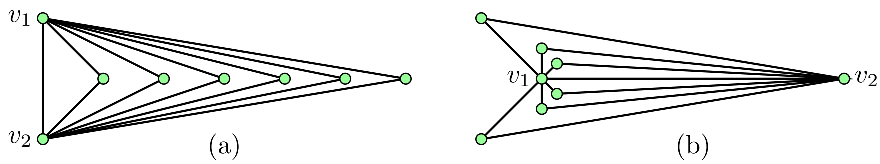


Figure 7: The graph $G_8$ in Lemma 5.

**Open Problem 2.** Does every $n$-vertex 2-tree have a (plane) drawing with at most $(2-\epsilon)n$ segments, for some $\epsilon > 0$? Note that the graph $G_n$ from Lemma 5 has a plane drawing with $3n/2-2$ segments, as illustrated in Figure 7(b).

We now turn our attention to drawings of planar 3-trees.

**Theorem 3.** *Every $n$-vertex plane 3-tree has a plane drawing with at most $2n-2$ segments. For all $n \geq 4$, there is an $n$-vertex plane 3-tree with at least $2n-2$ slopes (and thus at least $2n-2$ segments) in every drawing.*

*Proof.* We prove the upper bound by induction on $n$ with the hypothesis that "every plane 3-tree with $n \geq 4$ vertices has a plane drawing with at most $2n-2$ segments, such that for every internal face $F$ there is an edge $e$ incident to exactly one vertex of $F$, and the extension of $e$ intersects the interior of $F$." The base case is trivial since $K_4$ is the only 3-tree on four vertices, and any plane drawing of $K_4$ satisfies the hypothesis.

Suppose that the claim holds for plane 3-trees on $n-1$ vertices. Let $G$ be a plane 3-tree on $n$ vertices. Every $k$-tree on at least $k+2$ vertices has two non-adjacent simplicial vertices of degree exactly $k$ [13]. In particular, $G$ has two non-adjacent simplicial degree-3 vertices, one of which is not on the outerface. Thus $G$ can be obtained from $G \setminus v$ by adding $v$ inside some internal face $(p, q, r)$ of $G \setminus v$, adjacent to $p$, $q$ and $r$[¶]. By induction, $G \setminus v$ has a drawing with $2n-4$ segments in which there is an edge $e$ incident to exactly

[¶]This implies that the planar 3-trees are precisely those graphs that are produced by the LEDA 'random' maximal planar graph generator. This algorithm, starting from $K_3$, repeatedly adds a new vertex adjacent to the three vertices of a randomly selected internal face.

one of $\{p, q, r\}$, and the extension of $e$ intersects the interior of the face. Position $v$ in the interior of the face anywhere on the extension of $e$, and draw segments from $v$ to each of $p$, $q$ and $r$. We obtain a plane drawing of $G$ with $2n-2$ segments. The extension of $vp$ intersects the interior of $(v, q, r)$; the extension of $vq$ intersects the interior of $(v, p, r)$; and the extension of $vr$ intersects the interior of $(v, p, q)$. All other faces of $G$ are faces of $G \setminus v$. Thus the inductive hypothesis holds for $G$, and the proof of the upper bound is complete.

For each $n \geq 4$ we now provide a family $\mathcal{G}_n$ of $n$-vertex plane 3-trees, each of which require at least $2n-2$ segments in any drawing. Let $\mathcal{G}_4 = \{K_4\}$. Obviously every plane drawing of $K_4$ has six segments. For all $n \geq 5$, let $\mathcal{G}_n$ be the family of plane 3-trees $G$ obtained from some plane 3-tree $H \in \mathcal{G}_{n-1}$ by adding a new vertex $v$ in the outerface of $H$ adjacent to each of the three vertices of the outerface. Any drawing of $G$ contains a drawing of $H$, which contributes at least $2n-4$ segments by induction. In addition, the two edges incident to $v$ on the triangular outerface of $G$ are each in their own segment. Thus $G$ has at least $2n-2$ segments. □

## 4 3-Connected Plane Graphs

The following is the main result of this section.

**Theorem 4.** *Every 3-connected plane graph with $n$ vertices has a plane drawing with at most $5n/2-3$ segments and at most $2n-10$ slopes.*

The proof of Theorem 4 is based on the canonical ordering of Kant [31], which is a generalisation of a similar structure for plane triangulations introduced by de Fraysseix *et al.* [9]. Let $G$ be a 3-connected plane graph. Kant [31] proved that $G$ has a canonical ordering defined as follows. Let $\sigma = (V_1, V_2, \dots, V_K)$ be an ordered partition of $V(G)$. That is, $V_1 \cup V_2 \cup \cdots \cup V_K = V(G)$ and $V_i \cap V_j = \emptyset$ for all $i \neq j$. Define $G_i$ to be the plane subgraph of $G$ induced by $V_1 \cup V_2 \cup \cdots \cup V_i$. Let $C_i$ be the subgraph of $G$ induced by the edges on the boundary of the outerface of $G_i$. As illustrated in Figure 8, $\sigma$ is a *canonical ordering* of $G$ if:

- $V_1 = \{v_1, v_2\}$, where $v_1$ and $v_2$ lie on the outerface and $v_1v_2 \in E(G)$.
- $V_K = \{v_n\}$, where $v_n$ lies on the outerface, $v_1v_n \in E(G)$, and $v_n \neq v_2$.
- Each $C_i$ $(i > 1)$ is a cycle containing $v_1v_2$.
- Each $G_i$ is biconnected and internally 3-connected; that is, removing any two interior vertices of $G_i$ does not disconnect it.
- For each $i \in \{2, 3, \dots, K-1\}$, one of the following condition holds:
  1. $V_i = \{v_i\}$ where $v_i$ is a vertex of $C_i$ with at least three neighbours in $C_{i-1}$, and $v_i$ has at least one neighbour in $G \setminus G_i$.
  2. $V_i = (s_1, s_2, \dots, s_\ell, v_i)$, $\ell \geq 0$, is a path in $C_i$, where each vertex in $V_i$ has at least one neighbour in $G \setminus G_i$. Furthermore, the first and the last vertex in $V_i$ have one neighbour in $C_{i-1}$, and these are the only two edges between $V_i$ and $G_{i-1}$.

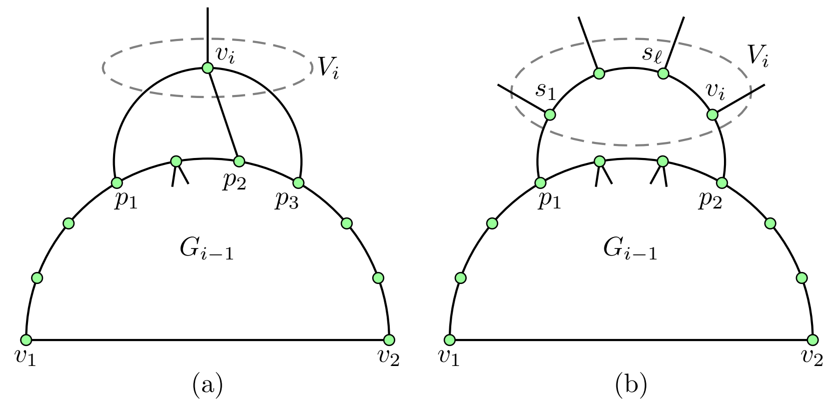


Figure 8: The canonical ordering of a 3-connected plane graph.

The vertex $v_i$ is called the *representative* vertex of the set $V_i$, $2 \leq i \leq K$. The vertices $\{s_1, s_2, \dots, s_\ell\} \subseteq V_i$ are called *division* vertices. Let $S \subset V(G)$ be the set of all division vertices. A vertex $u$ is a *successor* of a vertex $w \in V_i$ if $uw$ is an edge and $u \in G \setminus G_i$. A vertex $u$ is a *predecessor* of a vertex $w \in V_i$ if $uw$ is an edge and $u \in V_j$ for some $j < i$. We also say that $u$ is a predecessor of $V_i$. Let $P(V_i) = (p_1, p_2, \dots, p_q)$ denote the set of predecessors of $V_i$ ordered by the path from $v_1$ to $v_2$ in $C_{i-1} \setminus v_1v_2$. Vertex $p_1$ and $p_q$ are the *left* and *right predecessors* of $V_i$ respectively, and vertices $p_2, p_3, \dots p_{q-1}$ are called *middle predecessors* of $V_i$.

**Theorem 5.** *Let $G$ be an $n$-vertex $m$-edge plane 3-connected graph with a canonical ordering $\sigma$. Define $S$ as above. Then $G$ has a plane drawing $D$ with at most*

$$m - \max\{\lceil n/2 \rceil - |S| - 3,\ |S|\}$$

*segments, and at most*

$$m - \max\{n - |S| - 4,\ |S|\}$$

*slopes.*

*Proof.* We first define $D$ and then determine the upper bounds on the number of segments and slopes in $D$. For every vertex $v$, let $X(v)$ and $Y(v)$ denote the $x$ and $y$ coordinates of $v$, respectively. If a vertex $v$ has a neighbour $w$, such that $X(w) < X(v)$ and $Y(w) < Y(v)$, then we say $vw$ is a *left edge* of $v$. Similarly, if $v$ has a neighbour $w$, such that $X(w) > X(v)$ and $Y(w) < Y(v)$, then we say $vw$ is a *right edge* of $v$. If $vw$ is an edge such that $X(v) = X(w)$ and $Y(v) < Y(w)$, than we say $vw$ is a *vertical edge above* $v$ and *below* $w$.

We define $D$ inductively on $\sigma = (V_1, V_2, \dots, V_K)$ as follows. Let $D_i$ denote a drawing of $G_i$. A vertex $v$ is a *peak in* $D_i$, if each neighbour $w$ of $v$ has $Y(w) \leq Y(v)$ in $D_i$. We say that a point $p$ in the plane is *visible in* $D_i$ from vertex $v \in D_i$, if the segment $\overline{pv}$ does not intersect $D_i$ except at $v$. At the $i^{\text{th}}$ induction step, $2 \leq i \leq K$, $D_i$ will satisfy the following invariants:

**Invariant 1:** $C_i \setminus v_1v_2$ is *strictly X-monotone*; that is, the path from $v_1$ to $v_2$ in $C_i \setminus v_1v_2$ has increasing X-coordinates.

**Invariant 2:** Every peak in $D_i$, $i < K$, has a successor.

**Invariant 3:** Every representative vertex $v_j \in V_j$, $2 \leq j \leq i$ has a left and a right edge. Moreover, if $|P(V_j)| \geq 3$ then there is a vertical edge below $v_j$.

**Invariant 4:** $D_i$ has no edge crossings.

For the base case $i = 2$, position the vertices $v_1$, $v_2$ and $v_3$ at the corners of an equilateral triangle so that $X(v_1) < X(v_3) < X(v_2)$ and $Y(v_1) < Y(v_2) < Y(v_3)$. Draw the division vertices of $V_2$ on the segment $v_1v_3$. This drawing of $D_2$ satisfies all four invariants. Now suppose that we have a drawing of $D_{i-1}$ that satisfies the invariants. There are two cases to consider in the construction of $D_i$, corresponding to the two cases in the definition of the canonical ordering.

**Case 1.** $|P(V_i)| \geq 3$: If $v_i$ has a middle predecessor $v_j$ with $|P(V_j)| \geq 3$, let $w = v_j$. Otherwise let $w$ by any middle predecessor of $v_i$. Let $L$ be the open ray $\{(X(w), y) : y > Y(w)\}$. By invariant 1 for $D_{i-1}$, there is a point in $L$ that is visible in $D_{i-1}$ from every predecessor of $v_i$. Represent $v_i$ by such a point, and draw segments between $v_i$ and each of its predecessors. That the resulting drawing $D_i$ satisfies the four invariants can be immediately verified.

**Case 2.** $|P(V_i)| = 2$: Suppose that $P(V_i) = \{w, u\}$, where $w$ and $u$ are the left and the right predecessors of $V_i$, respectively. Suppose $Y(w) \geq Y(u)$. (The other case is symmetric.) Let $P$ be the path between $w$ and $u$ on $C_{i-1} \setminus v_1v_2$. As illustrated in Figure 9, let $A_i$ be the region $\{(x, y) : y > Y(w) \text{ and } X(w) \leq x \leq X(u)\}$.

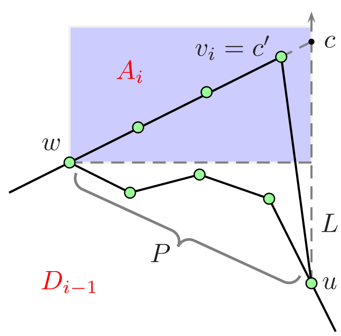


Figure 9: Illustration for Case 2.

Assume, for the sake of contradiction, that $D_{i-1} \cap A_i \neq \emptyset$. By the monotonicity of $D_{i-1}$, $P \cap A_i \neq \emptyset$. Let $p \in P \cap A_i$. Since $Y(p) > Y(w) \geq Y(u)$, $P$ is $X$-monotone and thus has a vertex between $w$ and $u$ that is a peak. By the definition of the canonical ordering $\sigma$, the addition of $V_i$ creates a face of $G$, since $V_i$ is added in the outerface of $G_{i-1}$. Therefore, each vertex between $w$ and $u$ on $P$ has no successor, and is thus not a peak in $D_{i-1}$ by invariant 2, which is the desired contradiction. Therefore $D_{i-1} \cap A_i = \emptyset$.

Let $L$ be the open ray $\{(X(u), y) : y > Y(u)\}$. If $w \notin S$, then by invariant 3, $w$ has a left and a right edge in $D_{i-1}$. Let $c$ be the point of intersection between $L$ and the line extending the left edge at $w$. If $w \in S$, then let $c$ be any point in $A_i$ on $L$. By invariant 1, there is a point $c' \notin \{c, w\}$ on $\overline{wc}$ such that $c'$ is visible in $D_{i-1}$ from $u$. Represent $v_i$ by $c'$, and draw two segments $\overline{v_iu}$ and $\overline{v_iw}$. These two segments do not intersect any part of $D_{i-1}$ (and neither is horizontal). Represent any division vertices in $V_i$ by arbitrary points on the open segment $\overline{wv_i} \cap A_i$. Therefore, in the resulting drawing $D_i$, there are no crossings and the remaining three invariants are maintained.

This completes the construction of $D$. The following claim will be used to bound the number of segments and slopes in $D$. It basically says that a division vertex (and $v_2$) can be the higher predecessor for at most one set $V_i$ with $|P(V_i)| = 2$.

**Claim 1.** *Let $V_i, V_j \in \sigma$ with $i < j$ and $|P(V_i)| = |P(V_j)| = 2$. Let $w_i$ be the higher of the two predecessors of $V_i$ in $D_{i-1}$, and let $w_j$ be the higher of the two predecessors of $V_j$ in $D_{j-1}$. If $w_i \in S$ or $w_i = v_2$, then $w_i \neq w_j$.*

*Proof.* Suppose that $w_i \in V_k$, $k < i$. First assume that $w_i \in S$. Then each division vertex lies on some non-horizontal segment and it is not an endpoint of that segment. Thus $w_i$ is not a peak in $D_k$, and therefore it is not a peak in every $D_\ell$, $\ell \geq k$. For all $\epsilon > 0$, let

$$A'_\epsilon = \{(x, y) \,:\, y > Y(w_i),\ X(w_i) - \epsilon \leq x < X(w_i)\}, \text{ and}$$
$$A''_\epsilon = \{(x, y) \,:\, y > Y(w_i),\ X(w_i) < x \leq X(w_i) + \epsilon\}\ .$$

Then for all small enough $\epsilon$, either $A'_\epsilon \cap D_k \neq \emptyset$ or $A''_\epsilon \cap D_k \neq \emptyset$. Without loss of generality, $A'_\epsilon \cap D_k = \emptyset$ and $A''_\epsilon \cap D_k \neq \emptyset$. Then at iteration $i > k$, the region $A_i$, as defined in Case 2 of the construction of $D_i$, contains $A'_\epsilon$ for all small enough $\epsilon$. Thus, $A'_\epsilon \cap D_i \neq \emptyset$ for all small enough $\epsilon$. Since $j \geq i+1$, $A'_\epsilon \cap D_{j-1} \neq \emptyset$ or $A''_\epsilon \cap D_{j-1} \neq \emptyset$ for all small enough $\epsilon$. Therefore, $w_i \neq w_j$ (since $V_j$ is drawn by Case 2 of the construction of $D_j$, where it is known that $A_j \cap D_{j-1} = \emptyset$). The case $w_i = v_2$ is the same, since the region $A''_\epsilon \cap D_i = \emptyset$, for every $\epsilon$ and every $1 \leq i \leq K$, so only region $A'_\epsilon$ is used, and thus the above argument applies. □

For the purpose of counting the number of segments and slopes in $D$ assume that we draw edge $v_1v_2$ at iteration step $i = 1$ and $G_2 \setminus v_1v_2$ at iteration $i = 2$. In every iteration $i$ of the construction, $2 \leq i \leq K$, at most $|P(V_i)|$ new segments and slopes are created. We call an iteration $i$ of the construction *segment-heavy* if the difference between the number of segments in $D_i$ and $D_{i-1}$ is exactly $|P(V_i)|$, and *slope-heavy* if the difference between the number of slopes in $D_i$ and $D_{i-1}$ is exactly $|P(V_i)|$. Let $h_s$ and $h_\ell$ denote the total number of segment-heavy and slope-heavy iterations, respectively. Then $D$ uses at most

$$1 + \sum_{i=2}^{K} (|P(V_i)| - 1) + h_s \tag{1}$$

segments, and at most

$$1 + \sum_{i=2}^{K} (|P(V_i)| - 1) + h_\ell \tag{2}$$

slopes.

We first express $\sum_{i=2}^{K} |P(V_i)|$ in terms of $m$ and $|S|$, and then establish an upper bound on $h_s$ and $h_\ell$. For $i \geq 2$, let $E_i$ denote the set of edges of $G_i$ with at least one endpoint in $V_i$, and let $\ell_i$ denote the number of division vertices in $V_i$. Then $m = 1 + \sum_{i=2}^{K} |E_i| = 1 + \sum_{i=2}^{K} (\ell_i + |P(V_i)|) = 1 + |S| + \sum_{i=2}^{K} |P(V_i)|$. Thus $\sum_{i=2}^{K} |P(V_i)| = m - |S| - 1$. Since the trivial upper bound for $h_s$ and $h_\ell$ is $K - 1$, and by (1) and (2), we have that $D$ uses at most $1 + \sum_{i=2}^{K} |P(V_i)| = 1 + m - |S| - 1 = m - |S|$ segments and slopes.

We now prove a tighter bound on $h_s$. Let $R$ denote the set of representative vertices of segment-heavy steps $i$ with $|P(V_i)| \geq 3$. Consider a step $i$ such that $|P(V_i)| \geq 3$. If $v_i$ has at least one predecessor $v_j$ with $|P(V_j)| \geq 3$, then $v_i$ is drawn on the line that extends the vertical edge below $v_j$, and thus step $i$ introduces at most $|P(V_i)|-1$ new segments and is not segment-heavy. Therefore, step $i$ is segment-heavy only if no middle predecessor $w$ of $v_i$ is in $R$. Thus for each segment-heavy step $i$ with $|P(V_i)| \geq 3$, there is a unique vertex $w \notin R$. In other words, for each vertex in $R$, there is a unique vertex in $V(G) \setminus R$. Thus $|R| \leq \lfloor n/2 \rfloor$. Since the number of segment-heavy steps $i$ with $|P(V_i)| \geq 3$ is equal to $|R|$, there is at most $\lfloor n/2 \rfloor$ such steps.

The remaining steps, those with $|P(V_i)| = 2$, introduce $|P(V_i)|$ segments only if the higher of the two predecessors of $V_i$ is in $S$ or is $v_2$. (It cannot be $v_1$, since $Y(v_1) < Y(v)$ for every vertex $v \neq v_1$.) By the above claim, there may be at most $|S|+1$ such segment-heavy steps. Therefore, $h_s \leq \lfloor n/2 \rfloor + |S| + 1$. By (1) and since $K = n-1-|S|$, $D$ has at most $m - \lceil n/2 \rceil + |S| + 3$ segments.

Finally, we bound $h_\ell$. There may be at most one slope-heavy step $i$ with $|P(v_i)| \geq 3$, since there is a vertical edge below every such vertex $v_i$ by invariant 3. As in the above case for segments, there may be at most $|S|+1$ slope-heavy steps $i$ with $|P(v_i)| = 2$. Therefore, $h_\ell \leq |S|+2$. By (2) and since $K = n-1-|S|$, we have that $D$ has at most $m-n+|S|+4$ slopes. □

*Proof of Theorem 4.* Whenever a set $V_i$ is added to $G_{i-1}$, at least $|V_i|-1$ edges that are not in $G$ can be added so that the resulting graph is planar. Thus $|S| = \sum_i(|V_i| - 1) \leq 3n-6-m$. Hence Theorem 5 implies that $G$ has a plane drawing with at most $m-n/2+|S|+3 \leq 5n/2-3$ segments, and at most $m-n+|S|-4 \leq 2n-10$ slopes. □

We now prove that the bound on the number of segments in Theorem 4 is tight.

**Lemma 6.** *For all $n \equiv 0 \pmod 3$, there is an $n$-vertex planar triangulation with maximum degree six that has at least $2n-6$ segments in every plane drawing, regardless of the choice of outerface.*

*Proof.* Consider the planar triangulation $G_k$ with vertex set $\{x_i, y_i, z_i : 1 \leq i \leq k\}$ and edge set $\{x_iy_i, y_iz_i, z_ix_i : 1 \leq i \leq k\} \cup \{x_ix_{i+1}, y_iy_{i+1}, z_iz_{i+1} : 1 \leq i \leq k-1\} \cup \{x_iy_{i+1}, y_iz_{i+1}, z_ix_{i+1} : 1 \leq i \leq k-1\}$. $G_k$ has $n = 3k$ vertices. $G_k$ is the famous 'nested-triangles' graph. We say $\{(x_i, y_i, z_i) : 1 \leq i \leq k\}$ are *the triangles* of $G_k$. This graph has a natural plane embedding with the triangle $x_iy_iz_i$ nested inside the triangle $(x_{i+1}, y_{i+1}, z_{i+1})$ for all $1 \leq i \leq k-1$, as illustrated in Figure 10.

We first prove that if $(x_k, y_k, z_k)$ is the outerface then $G_k$ has at least $6k$ segments in any plane drawing. First observe that no two edges in the triangles can share a segment. Thus they contribute $3k$ segments.

We claim that the six edges between triangles $(x_i, y_i, z_i)$ and $(x_{i+1}, y_{i+1}, z_{i+1})$ contribute a further three segments. Consider the two edges $x_ix_{i+1}$ and $z_ix_{i+1}$ incident on $x_{i+1}$. We will show that at least one of them contributes a new segment. Let $R_x$ be the region bounded by the lines containing $x_iy_i$ and $x_iz_i$ that shares only $x_i$ with triangle $(x_i, y_i, z_i)$. Similarly, let $R_z$ be the region bounded by the lines containing $x_iz_i$ and $y_iz_i$ that shares only $z_i$ with the same triangle. We note that these two regions are disjoint. Furthermore, if edge $x_ix_{i+1}$ belongs to a segment including edges contained in triangle

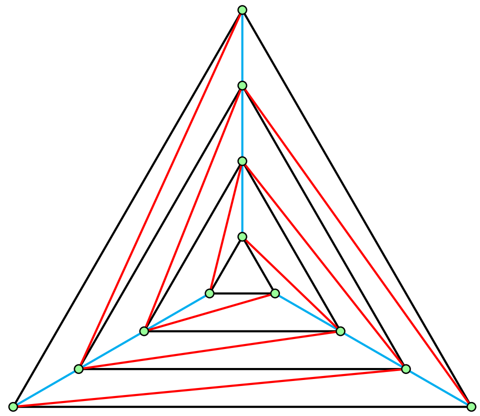

Figure 10: The graph $G_4$ in Lemma 6.

$(x_i, y_i, z_i)$, then $x_{i+1}$ lies in region $R_x$. Similarly, if $z_i x_{i+1}$ belongs to a segment including edges contained in triangle $(x_i, y_i, z_i)$, then $x_{i+1}$ lies in region $R_z$. Both cases cannot be true simultaneously so either edge $x_i x_{i+1}$ or edge $z_i x_{i+1}$ contributes a new segment to the drawing. Symmetric arguments apply the edges incident on $y_{i+1}$ and $z_{i+1}$ so the edges between triangles contribute at least three segments.

Thus in total we have at least $3k+3(k-1) = 2n-3$ segments. Now suppose that some face, other than $(x_k, y_k, z_k)$, is the outerface. Thus the triangles are split into two nested sets. Say there are $p$ triangles in one set and $q$ in the other. By the above argument, any drawing has at least $(2p-3)+(2q-3) = 2n-6$ segments. □

We now prove that the bound on the number of slopes in Theorem 4 is tight up to an additive constant.

**Lemma 7.** *For all $n \geq 3$, there is an $n$-vertex planar triangulation $G_n$ that has at least $n+2$ slopes in every plane drawing. For a particular choice of outerface, there are at least $2n-2$ slopes in every plane drawing.*

*Proof.* Let $G_n$ be the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1 v_i, v_2 v_i : 3 \leq i \leq n\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. $G_n$ is a planar triangulation. Every 3-cycle in $G_n$ contains $v_1$ or $v_2$. Thus $v_1$ or $v_2$ is in the boundary of the outerface in every plane drawing of $G_n$. By Lemma 1, the number of slopes in any plane drawing of $G_n$ is at least $(n-3)+1+1+3 = n+2$. As illustrated in Figure 11(a), if we fix the outerface of $G_n$ to be $(v_1, v_2, v_n)$, then the number of slopes is at least $(n-3)+(n-3)+1+3 = 2n-2$ slopes by Lemma 1 □

As illustrated in Figure 11(b), the graph $G_n$ in Lemma 7 has a plane drawing (using a different embedding) with only $\lceil 3n/2 \rceil$ slopes.

Since deleting an edge from a drawing cannot increase the number of slopes, and every plane graph can be triangulated to a 3-connected plane graph, Theorem 4 implies:

**Corollary 2.** *Every $n$-vertex plane graph has a plane drawing with at most $2n-10$ slopes.* □

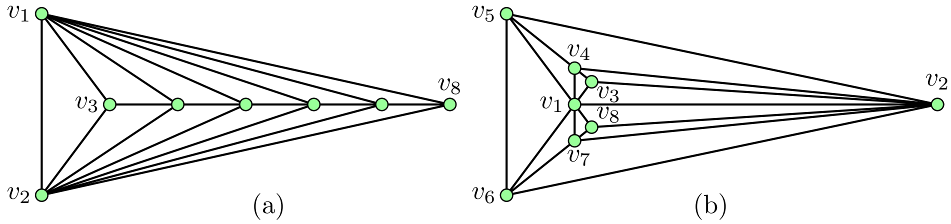


Figure 11: The graph $G_8$ from Lemma 7.

**Open Problem 3.** Is there some $\epsilon > 0$, such that every $n$-vertex planar triangulation has a plane drawing with $(2-\epsilon)n + \mathcal{O}(1)$ slopes?

On the other hand, Theorem 4 does not imply any upper bound on the number of segments for all planar graphs. A natural question to ask is whether Theorem 4 can be extended to plane graphs that are not 3-connected. We have the following lower bound.

**Lemma 8.** *For all even $n \geq 4$, there is a 2-connected plane graph with $n$ vertices (and $5n/2-4$ edges) that has as many segments as edges in every drawing.*

*Proof.* Let $G_n$ be the graph with vertex set $\{v, w, x_i, y_i : 1 \leq i \leq (n-2)/2\}$ and edge set $\{vw, x_iy_i, vx_i, vy_i, wx_i, wy_i : 1 \leq i \leq (n-2)/2\}$. Consider the plane embedding of $G_n$ with the cycle $(v, w, y_n)$ as the outerface, as illustrated in Figure 12. Since the outerface is a triangle, no two edges incident to $v$ can share a segment, and no two edges incident to $w$ can share a segment. Consider two edges $e$ and $f$ both incident to a vertex $x_i$ or $y_i$. The endpoints of $e$ and $f$ induce a triangle. Thus $e$ and $f$ cannot share a segment. Therefore no two edges in $G_n$ share a segment. □

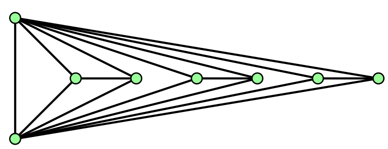

Figure 12: The graph $G_8$ in Lemma 8.

Note that the drawing technique from Figure 7 can be used to draw the graph $G_n$ in Lemma 8 with only $2n + \mathcal{O}(1)$ segments.

**Open Problem 4.** What is the minimum $c$ such that every $n$-vertex plane (or planar) graph has a plane drawing with at most $cn + \mathcal{O}(1)$ segments?

### 4.1 Cubic 3-Connected Plane Graphs

A graph in which every vertex has degree three is *cubic*. It is easily seen that Theorem 5 implies that every cubic plane 3-connected graph on $n$ vertices has a plane drawing with at most $5n/4 + \mathcal{O}(1)$ segments. This result can be improved as follows.

**Lemma 9.** *Every cubic plane 3-connected graph $G$ on $n$ vertices has a plane drawing with at most $n+2$ segments.*

*Proof.* Let $D$ be the plane drawing of $G$ from Theorem 5. Recall the definitions and the arguments for counting segments in Theorem 5. By (1), the number of segments is at most

$$1 + h_s + \sum_{i=2}^{K} (|P(V_i)| - 1) \ .$$

By the properties of the canonical ordering for plane cubic graphs, $|P(V_i)| = 2$ for all $2 \leq i \leq K - 1$, and $|P(V_K)| = 3$. Thus $|R| \leq 1$. As in Theorem 5, the number of segment-heavy steps with $|P(V_i)| = 2$ is at most $|S| + 1$. Thus $h_s \leq |S| + 2$. Therefore the number of segments in $D$ is at most

$$1 + (|S| + 2) + (K - 2) + 2 \ = \ |S| + 3 + K \ = \ |S| + 3 + n - 1 - |S| \ = \ n + 2 \ ,$$

as claimed. □

Our bound on the number of slopes in a drawing of a 3-connected plane graph (Theorem 4) can be drastically improved when the graph is cubic.

**Theorem 6.** *Every cubic* 3*-connected plane graph has a plane drawing in which every edge has slope in* $\{\pi/4, \pi/2, 3\pi/4\}$*, except for three edges on the outerface.*

*Proof.* Let $\sigma = (V_1, V_2, \dots, V_K)$ be a canonical ordering of $G$. We re-use the notation from Theorem 5, except that a representative vertex of $V_i$ may be the first or last vertex in $V_i$. Since $G$ is cubic, $|P(V_i)| = 2$ for all $1 < i < K$, and every vertex not in $\{v_1, v_2, v_n\}$ has exactly one successor. We proceed by induction on $i$ with the hypothesis that $G_i$ has a plane drawing $D_i$ that satisfies the following invariants.

**Invariant 1:** $C_i \setminus v_1v_2$ is $X$*-monotone*; that is, the path from $v_1$ to $v_2$ in $C_i \setminus v_1v_2$ has non-decreasing X-coordinates.

**Invariant 2:** Every peak in $D_i$, $i < K$, has a successor.

**Invariant 3:** If there is a vertical edge above $v$ in $D_i$, then all the edges of $G$ that are incident to $v$ are in $G_i$.

**Invariant 4:** $D_i$ has no edge crossings.

Let $D_2$ be the drawing of $G_2$ constructed as follows. Draw $v_1v_2$ horizontally with $X(v_1) < X(v_2)$. This accounts for one edge whose slope is not in $\{\pi/4, \pi/2, 3\pi/4\}$. Now draw $v_1v_3$ with slope $\pi/4$, and draw $v_2v_3$ with slope $3\pi/4$. Add any division vertices on the segment $v_1v_3$. Now $v_3$ is the only peak in $D_2$, and it has a successor by the definition of the canonical ordering. Thus all the invariants are satisfied for the base case $D_2$.

Now suppose that $2 < i < K$ and we have a drawing of $D_{i-1}$ that satisfies the invariants. Suppose that $P(V_i) = \{u, w\}$, where $u$ and $w$ are the left and the right predecessors of $V_i$, respectively. Without loss of generality, $Y(w) \leq Y(u)$. Let the representative vertex $v_i$ be last vertex in $V_i$. Position $v_i$ at the intersection of a vertical segment above $w$, and a segment of slope $\pi/4$ from $u$, and add any division vertices on $\overline{uv_i}$, as illustrated in Figure 13(a). Note that there is no vertical edge above $w$ by invariant 3 for $D_{i-1}$. (For

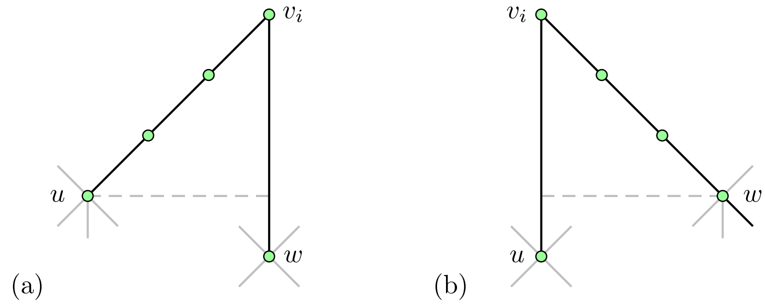


Figure 13: Construction of a 3-slope drawing of a cubic 3-connected plane graph.

the case in which $Y(u) < Y(w)$, we take the representative vertex $v_i$ to be the first vertex in $V_i$, and the edge $wv_i$ has slope $3\pi/4$, as illustrated in Figure 13(b).)

Clearly the resulting drawing $D_i$ is $X$-monotone. Thus invariant 1 is maintained. The vertex $v_i$ is the only peak in $D_i$ that is not a peak in $D_{i-1}$. Since $v_i$ has a successor by the definition of the canonical ordering, invariant 2 is maintained. The vertical edge $wv_i$ satisfies invariant 3, since $v_i$ is the sole successor of $w$. Thus invariant 3 is maintained. No vertex between $u$ and $w$ (on the path from $u$ to $w$ in $C_{i-1} \setminus v_1v_2$) is higher than the higher of $u$ and $w$. Otherwise there would be a peak, not equal to $v_n$, with no successor, and thus violating invariant 2 for $D_{i-1}$. Thus the edges in $D_i \setminus D_{i-1}$ do not cross any edges in $D_i$. In particular, there is no edge $ux$ in $D_{i-1}$ with slope $\pi/4$ and $Y(x) > Y(u)$.

It remains to draw the vertex $v_n$. Suppose $v_n$ is adjacent to $v_1$, $u$, and $w$, where $X(v_1) < X(u) < X(w)$. By invariants 1 and 3 applied to $v_1$, $u$ and $w$, there is point $p$ vertically above $u$ that is visible from $v_1$ and $w$. Position $v_n$ at $p$ and draw its incident edges. We obtain the desired drawing of $G$. The edge $v_nu$ has slope $\pi/2$, while $v_nv_1$ and $v_nw$ are the remaining two edges whose slope is not in $\{\pi/4, \pi/2, 3\pi/4\}$. □

Note that in Theorem 6 we could have chosen any set of three slopes instead of $\{\pi/4, \pi/2, 3\pi/4\}$. By Lemma 1, the bound of six on the number of slopes in Theorem 6 is optimal for any 3-connected cubic plane graph whose outerface is a triangle. It is easily seen that there is such a graph on $n$ vertices for all even $n \geq 4$.

**Corollary 3.** *Every cubic* 3*-connected plane graph has a plane 'drawing' with three slopes and three bends on the outerface.*

*Proof.* Apply the proof of Theorem 6 with two exceptions. First the edge $v_1v_2$ is drawn with one bend. The segment incident to $v_1$ has slope $3\pi/4$, and the segment incident to $v_2$ has slope $\pi/4$. The second exception regards how to draw the edges incident to $v_n$. Suppose $v_n$ is adjacent to $v_1$, $u$, and $w$, where $X(v_1) < X(u) < X(w)$. There is a point $s$ above $v_1$, a point $p$ above $u$, and a point $t$ above $w$, so that the slope of $sp$ is $\pi/4$ and the slope of $tp$ is $3\pi/4$. Place $v_n$ at $p$, draw the edge $v_nu$ vertical, draw the edge $v_1v_n$ with one bend through $s$ (with slopes $\{\pi/2, \pi/4\}$), and draw the edge $wv_n$ with one bend through $t$ (with slopes $\{\pi/2, 3\pi/4\}$). □

**Open Problem 5.** Does there exist a function $f$ such that every plane graph with maximum degree $\Delta$ has a plane drawing with $f(\Delta)$ slopes? This is open even for maximal outerplanar graphs.

## 5 Complete Multipartite Graphs

In a drawing of the complete graph $K_n$, no three vertices are collinear. Thus the number of segments is $\binom{n}{2}$. The answer for slopes is more interesting and well known.

**Lemma 10.** *The minimum number of slopes in a drawing of the complete graph $K_n$ is $n$.*

*Proof.* The lower bound of $n$ follows from Lemma 1 by taking any three consecutive vertices on the convex hull. More generally, Jamison [29] proved that if a drawing of $K_n$ has $k$ vertices on the convex hull then the number of slopes is at least $k(n-2)/(k-2)$.

For the upper bound, consider a drawing of a graph $G$ on a regular $n$-gon with vertex ordering $(v_1, v_2, \ldots, v_n)$. Scott [46] observed that the number of slopes is

$$|\{(i+j) \bmod n : v_iv_j \in E(G)\}| \ . \tag{3}$$

Thus for $K_n$ on a regular $n$-gon, the number of slopes is $n$, as illustrated in Figure 14. In fact, Jamison [29] proved that every drawing of $K_n$ with exactly $n$ slopes is affinely equivalent to a regular $n$-gon. □

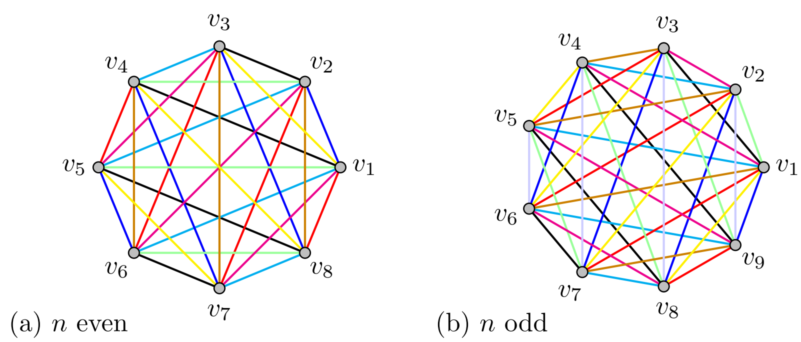


Figure 14: Drawings of $K_n$ with $n$ slopes.

Note that Wade and Chu [52], seemingly unaware of the earlier work of Scott and Jamison, rediscovered that the minimum number of slopes in a drawing of the complete graph $K_n$ is $n$. They also presented an algorithm to test if $K_n$ can be drawn using a given set of slopes.

**Lemma 11.** *The minimum number of slopes in a drawing of $K_{n,n}$ is $n$.*

*Proof.* Since $K_{n,n}$ is $n$-regular, the number of slopes in any drawing of $K_{n,n}$ is at least $n$ by Lemma 3(b). For the upper bound, position the vertices of $K_{n,n}$ on a regular $2n$-gon $(v_1, v_2, \ldots, v_{2n})$, alternating between the colour classes, as illustrated in Figure 15. Thus $v_iv_j$ is an edge if and only if $i+j$ is odd. By (3), the number of slopes is $|\{(i+j) \bmod 2n : 1 \leq i < j \leq 2n, i+j \text{ is odd}\}| = n$. □

**Lemma 12.** *The minimum number of slopes in a convex drawing of the complete bipartite graph $K_{a,b}$ is* $\max\{a, b\}$.

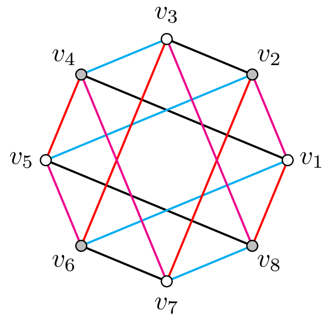


Figure 15: Drawing of $K_{4,4}$ with 4 slopes.

*Proof.* Without loss of generality $a \leq b$. Then $K_{a,b}$ has maximum degree $b$, and the number of slopes in a convex drawing is at least $b$ by Lemma 3(c). The upper bound follows from Lemma 11 since $K_{a,b} \subseteq K_{b,b}$. □

Lemma 12 is not necessarily optimal for non-convex drawings.

**Lemma 13.** *Every complete bipartite graph $K_{a,b}$ has a drawing with $\lceil b/2 \rceil + a - 1$ slopes.*

*Proof.* Without loss of generality $a \leq b$ and $b$ is even. Suppose $V(K_{a,b}) = \{v_1, v_2, \dots, v_a\} \cup \{u_1, u_2, \dots, u_{b/2}\} \cup \{w_1, w_2, \dots, w_{b/2}\}$, and $E(K_{a,b}) = \{v_i u_j, v_i w_j : 1 \leq i \leq a, 1 \leq j \leq b/2\}$. Position each vertex $u_j$ at $(j, 1)$; position each vertex $v_i$ at $(b/2 + i, 0)$; and position each vertex $w_j$ at $(b/2 + a + j, -1)$. Then every edge is parallel with one of the $b/2 + a - 1$ edges $\{v_1 u_j : 1 \leq j \leq b/2\} \cup \{u_1 v_i : 2 \leq i \leq a\}$, as illustrated in Figure 16. □

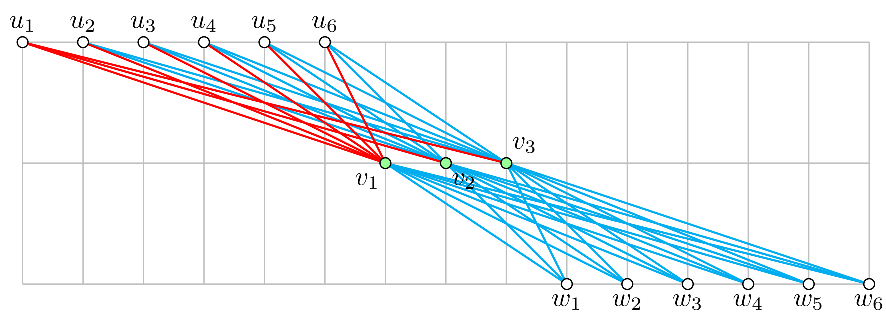


Figure 16: Drawing of $K_{3,12}$ with 8 slopes (highlighted).

Whether every complete multipartite graph with maximum degree $\Delta$ has a convex drawing with $\Delta + \mathcal{O}(1)$ slopes is an interesting open problem. We have the following partial solution.

**Lemma 14.** *Given integers $p \geq 0$ and $k \geq 2$, where $k - 1$ is a power of two, let $G$ be the complete $k$-partite graph $G = K_{2^p, 2^p, 2^{p+1}, \dots, 2^{p+1}}$. Then $G$ has a convex drawing with $\Delta(G)$ slopes.*

*Proof.* Let $n = (k-1)2^{p+1}$ be the number of vertices in $G$. Note that $n$ is a power of two, and $\Delta(G) = n - 2^p$. Let $V(G) = \{0, 1, \dots, n-1\}$. In what follows $a \equiv b$ means that $a \equiv b \pmod{n/2^p}$, and $a \equiv \pm b$ means that $a \equiv b$ or $a \equiv -b$. For all $0 \leq i \leq k-1$, let

$P_i = \{j \in V(G) : i \equiv \pm j\}$. Below we prove that $\{P_0, P_1, \ldots, P_{k-1}\}$ is a partition of $V(G)$ with $|P_0| = |P_{k-1}| = 2^p$, and $|P_i| = 2^{p+1}$ for all $1 \leq i \leq k-2$. Thus $\{P_0, P_1, \ldots, P_{k-1}\}$ defines a valid assignment of the vertices to the colour classes. To obtain the drawing of $G$, place the vertices in numerical order on the vertices of a regular $n$-gon.

For each vertex $j \in V(G)$, let $j' = j \bmod n/2^p$. If $0 \leq j' \leq n/2^{p+1}$, then $j \in P_{j'}$. Otherwise, $n/2^{p+1} < j' < n/2^p$, and $j \in P_{n/2^p - j'}$. Thus, each vertex belongs to at least one $P_i$. Suppose that $j \in P_i \cap P_h$. Thus $i \equiv \pm j$ and $h \equiv \pm j$, implying $i \equiv \pm h$. Since $0 \leq i \leq n/2^{p+1}$, we have $h = i$. Thus, each vertex belongs to exactly one $P_i$, and $\{P_0, P_1, \ldots, P_{k-1}\}$ is a partition of $V(G)$. The set $P_0$ has size $2^p$ because it is the set of all multiples of $n/2^p$ in $\{0, 1, \ldots, n-1\}$. Similarly, $P_{k-1}$ has size $2^p$ because it is the set of all odd multiples of $n/2^{p+1}$ in $\{0, 1, \ldots, n-1\}$. The remainder of the $P_i$'s have the same size, $2^{p+1}$, by symmetry.

To prove that the number of slopes $|\{(i+j) \bmod n : ij \in E(G)\}| = n - 2^p$, by (3), it suffices to prove that $i + j \equiv 0$ implies $ij \notin E(G)$. Suppose that $i \in P_h$. Thus $h + i \equiv 0$ or $h - i \equiv 0$. In the first case, we have $h + i \equiv i + j$, implying $h - j \equiv 0$. In the second case, we have $h - i + (i + j) \equiv 0$, implying $h + j \equiv 0$. In both cases $j \in P_h$, implying $ij \notin E(G)$. □

**Corollary 4.** *Given integers $p \geq 0$, $q \leq 2^p$, and $k \geq 2$, where $k-1$ is a power of two, let $G$ be the complete $k$-partite graph $G = K_{q,2^p,2^{p+1},\ldots,2^{p+1}}$. Then $G$ has a convex drawing with $\Delta(G)$ slopes.*

*Proof.* Let $G'$ be the complete $k$-partite graph $K_{2^p,2^p,2^{p+1},\ldots,2^{p+1}}$. Then $G$ is a subgraph of $G'$, and $\Delta(G) = \Delta(G') = (k-2)2^{p+1} + 2^p$. The result follows from Lemma 14. □

# 6 Drawings of General Graphs with Few Slopes

The following is the fundamental open problem regarding graph drawings with few slopes.

**Open Problem 6.** Is there a function $f$ such that every graph with maximum degree $\Delta$ has a drawing with at most $f(\Delta)$ slopes? This is open even for $\Delta = 3$.

A number of comments regarding Open Problem 6 are in order:

- The best lower bound that we are aware of is $\Delta + 1$ for the complete graph.
- There is no such function $f$ for convex drawings. Malitz [32] proved that there are $\Delta$-regular $n$-vertex graphs with book thickness $\Omega(\sqrt{\Delta} n^{1/2-1/\Delta})$. Since book thickness is a lower bound on the number of slopes in a convex drawing, every convex drawing of such a graph has $\Omega(\sqrt{\Delta} n^{1/2-1/\Delta})$ slopes.
- An affirmative solution to Open Problem 6 would imply that geometric thickness is bounded by maximum degree, which is an open problem due to Eppstein [19]. Duncan *et al.* [16] recently proved that graphs with maximum degree at most four have geometric thickness at most two.

In this section we provide an affirmative solution to Open Problem 6 for a class of intersection graphs that includes interval graphs and permutation graphs. For graphs of bounded degree and bounded treewidth we prove a $\mathcal{O}(\log n)$ bound on the number of slopes.

Our results are based on the following structure. Let $H$ be a (*host*) graph. The vertices of $H$ are called *nodes*. An *$H$-partition* of a graph $G$ is a function $f : V(G) \to V(H)$ such that for every edge $vw \in E(G)$ we have $f(v) = f(w)$ or $f(v)f(w) \in E(H)$. In the latter case, we say $vw$ is *mapped* to the edge $f(v)f(w)$. The *width* of $f$ is the maximum of $|f^{-1}(x)|$, taken over all nodes $x \in V(H)$, where $f^{-1}(x) = \{v \in V(G) : f(v) = x\}|$. The following general result describes how to produce a drawing of a graph $G$ given an $H$-partition of $G$ and a drawing of $H$.

**Lemma 15.** *Let $H$ be a graph admitting a drawing $D$ with $s$ distinct slopes and $\ell$ distinct edge lengths. Let $G$ be a graph admitting an $H$-partition of width $k$. Then $G$ has a drawing with $ks\ell(k-1) + k + s$ slopes.*

*Proof.* The general approach is to scale $D$ appropriately, and then replace each node of $H$ by a copy of the drawing of $K_k$ on a regular $k$-gon given in Lemma 10. The only difficulty is to scale $D$ so that we obtain a valid drawing of $G$.

Let $\{\theta_1, \theta_2, \dots, \theta_s\}$ be the set of slopes of the edges of $D$. Let $\{\beta_1, \beta_2, \dots, \beta_k\}$ be the set of slopes of the edges in the drawing of $K_k$ on a regular $k$-gon given in Lemma 10. Let $\angle(\phi_1, \phi_2)$ denote the size in radians of the minimum angle formed by lines of slope $\phi_1$ and $\phi_2$. Let $\epsilon = \min\{\angle(\theta_i, \beta_j)/2 : 1 \leq i \leq s, 1 \leq j \leq k, \theta_i \neq \beta_j\}$. Now rotate the drawing of $K_k$ by $\epsilon$ radians in an arbitrary direction, and recompute the $\beta_j$'s. Thus $\angle(\theta_i, \beta_j) \geq \epsilon$ for all $i$ and $j$.

Replace each node $x$ in $D$ by a disc $B_x$ of uniform radius $r$ centred at $x$, where $r$ is chosen small enough so that: (1) $B_x \cap B_y = \emptyset$ for all distinct nodes $x$ and $y$ in $D$; and (2) for every edge $xy \in E(H)$ with slope $\theta_i$, every segment with endpoints in $B_x$ and $B_y$ and with slope $\phi$ intersects no other $B_z$, and $\angle(\phi, \theta_i) < \epsilon$. Position a regular $k$-gon on each $B_x$ (using the orientation determined above), and position the vertices $f^{-1}(x)$ of $G$ at its vertices. Since $\angle(\theta_i, \beta_j) \geq \epsilon$, the slope of any edge $vw$ of $G$ that is mapped to $xy$ does not equal any $\beta_j$. Hence $vw$ does not pass through any other vertex of $G$.

Each copy of $K_k$ contributes the same $k$ slopes to the drawing of $G$. For each edge $xy \in E(H)$, for all $1 \leq i \leq k$, the edge of $G$ from the $i^{\text{th}}$ vertex on $B_x$ to the $i^{\text{th}}$ vertex on $B_y$ (if it exists) has the same slope as the edge $xy$ in $D$. Thus these edges contribute $s$ slopes to the drawing of $G$. Consider two edges $e_1$ and $e_2$ of $H$ that have the same slope and the same length in $D$. The edges of $G$ that are mapped to $e_1$ use the same set of slopes as the edges of $G$ that are mapped to $e_2$. There are at most $k^2 - k$ edges of $G$ that are mapped to a single edge of $H$ and were not counted above. Thus in total we have at most $k + s + s\ell(k^2 - k)$ slopes. □

## 6.1 Drawings Based on Paths

Lemma 15 suggests looking at host graphs that admit drawings with few slopes and few edge lengths. Obviously a path has a drawing with one slope and one edge length. The *path-partition-width* of a graph $G$, denoted by $\mathsf{ppw}(G)$, is the minimum $k$ such that $G$ has a $P$-partition of width $k$, for some path $P$. Lemma 15 with $s = \ell = 1$ implies:

**Corollary 5.** *Every graph $G$ has a drawing with $\mathsf{ppw}(G)^2 + 1$ slopes.* □

Path-partition-width is closely related to the classical graph parameter bandwidth. The *width* of a vertex ordering $(v_1, v_2, \dots, v_n)$ of a graph $G$ is the maximum of $|i-j|$, taken over all edges $v_iv_j \in E(G)$. The *bandwidth* of $G$, denoted by $\mathsf{bw}(G)$, is the minimum width of a vertex ordering of $G$.

**Lemma 16.** *For every graph $G$, $\frac{1}{2}(\mathsf{bw}(G)+1) \leq \mathsf{ppw}(G) \leq \mathsf{bw}(G)$.*

*Proof.* Let $(v_1, v_2, \dots, v_n)$ be a vertex ordering of $G$ with width $b = \mathsf{bw}(G)$. For all $0 \leq i \leq \lfloor n/b \rfloor$, let $B_i = \{v_{ib+1}, v_{ib+2}, \dots, v_{ib+b}\}$. Then $(B_0, B_1, \dots, B_{\lfloor n/b \rfloor})$ defines a path-partition of $G$ with width $b$. Thus $\mathsf{ppw}(G) \leq \mathsf{bw}(G)$.

Now suppose $(B_1, B_2, \dots, B_m)$ is a path-partition of $G$ with width $k = \mathsf{ppw}(G)$. Let $(v_1, v_2, \dots, v_n)$ be a vertex ordering of $G$ such that $i < j$ whenever $v_i \in B_p$ and $v_j \in B_q$ and $p < q$. For every edge $v_iv_j \in E(G)$ with $v_i \in B_p$ and $v_j \in B_q$, we have $|p-q| \leq 1$. Thus $|i-j| \leq 2k-1$. Hence the width of $(v_1, v_2, \dots, v_n)$ is at most $2k-1$. Therefore $\mathsf{bw}(G) \leq 2\mathsf{ppw}(G) - 1$. $\square$

Corollary 5 and Lemma 16 imply that every graph $G$ has a drawing with $\mathsf{bw}(G)^2 + 1$ slopes. This bound can be tweaked as follows.

**Theorem 7.** *Every graph $G$ has a drawing with at most $\frac{1}{2}\mathsf{bw}(G)\,(\mathsf{bw}(G)+1)+1$ slopes.*

*Proof.* Observe that in the construction of the path-partition in the proof of Lemma 16, the edges of each $G[B_i, B_{i+1}]$ are a subset of $\{\{v_{ib+j}, v_{(i+1)b+\ell}\} : 1 \leq j \leq b, 1 \leq \ell \leq j\}$. If we consistently assign the vertices in each $B_i$ to the regular $b$-gon in Lemma 15, then each $G[B_i, B_{i+1}]$ will use the same set of slopes, since each $G[B_i, B_{i+1}]$ is a subgraph of the same graph. The number of slopes in $G[B_i, B_{i+1}]$ is $1 + \sum_{j=1}^{b}(j-1)$, since each vertex $v_j \in B_i$ is incident to $j$ edges with endpoints in $B_{i+1}$, one of which is horizontal. Thus the total number of slopes in the resulting drawing of $G$ is $b+1+\frac{1}{2}(b-1)b = \frac{1}{2}b(b+1)+1$. $\square$

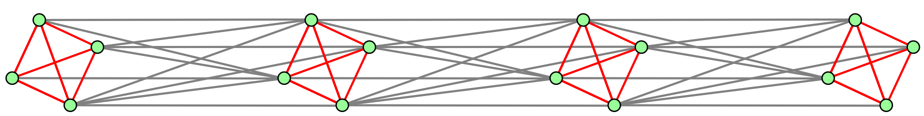

Figure 17: Drawing of a graph with bandwidth four with eleven slopes.

The following examples of Theorem 7 are corollaries of results by Fomin and Golovach [22] and Wood [54] that bound bandwidth in terms of maximum degree.

- Every interval graph $G$ has $\mathsf{bw}(G) \leq \Delta(G)$ [22, 54], and thus has a drawing with at most $\frac{1}{2}\Delta(G)\,(\Delta(G)+1)+1$ slopes.
- Every co-comparability graph $G$ (which includes the permutation graphs) has $\mathsf{bw}(G) \leq 2\Delta(G) - 1$ [54], and thus has a drawing with at most $\Delta(G)\,(2\Delta(G)-1)+1$ slopes.
- Every AT-free graph $G$ has $\mathsf{bw}(G) \leq 3\Delta(G)$ [54], and thus has a drawing with at most $\frac{3}{2}\Delta(G)\,(3\Delta(G)+1)+1$ slopes.

## 6.2 Drawings Based on Trees

A $T$-partition for some tree $T$ is called a *tree-partition*. Tree partitions have been extensively studied [4, 11, 12, 14, 25, 47]. Thus, Lemma 15 motivates the study of drawings of trees with few slopes and few distinct edge lengths.

**Lemma 17.** *Every tree $T$ with pathwidth $k \geq 1$ has a plane drawing with* $\max\{\Delta(T)-1, 1\}$ *slopes and $2k-1$ distinct edge lengths.*

The proof of Lemma 17 is loosely based on an algorithm of Suderman [48] for drawing trees on layers. We will need the following lemma.

**Lemma 18 ([48]).** *Every tree $T$ has a path $P$ such that $T \setminus V(P)$ has smaller pathwidth than $T$, and the endpoints of $P$ are leaves of $T$.*

*Proof.* Say $T$ has pathwidth $k$. Let $P$ be any path whose endpoints are in the first and last bag of a path decomposition of $T$ with width $k$. Then $P$ contains a vertex in every bag, and $T \setminus V(P)$ has pathwidth at most $k-1$. Obviously $P$ can be extended until its endpoints are leaves. □

A path $P$ satisfying Lemma 18 is called a *backbone* of $T$.

*Proof of Lemma 17.* We refer to $T$ as $T_0$. Let $n_0$ be the number of vertices in $T_0$, and let $\Delta_0 = \Delta(T_0)$. The result holds trivially for $\Delta_0 \leq 2$. Now assume that $\Delta_0 \geq 3$. Let $S$ be the set of slopes

$$S = \left\{ \frac{\pi}{2}\left(1 + \frac{i}{\Delta_0 - 2}\right) : 0 \leq i \leq \Delta_0 - 2 \right\} .$$

We proceed by induction on $n$ with the hypothesis: "There is real number $\ell = \ell(n_0, \Delta_0)$, such that for every tree $T$ with $n \leq n_0$ vertices, maximum degree at most $\Delta_0$, and pathwidth $k \geq 1$, and for every vertex $r$ of $T$ with degree less than $\Delta_0$, $T$ has a plane drawing $D$ in which:

- $r$ is at the top of $D$ (that is, no point in $D$ has greater Y-coordinate than $r$),
- every edge of $T$ has slope in $S$,
- every edge of $T$ has length in $\{\ell^0, \ell^1, \dots, \ell^{2k-1}\}$, and
- if $r$ is contained in some backbone of $T$, then every edge of $T$ has length in $\{\ell^0, \ell^1, \dots, \ell^{2k-2}\}$."

The result follows from the induction hypothesis, since we can take $r$ to be the endpoint of a backbone of $T_0$, in which case $\deg(r) = 1 < \Delta_0$, and thus every edge of $T_0$ has length in $\{\ell^0, \ell^1, \dots, \ell^{2k-2}\}$.

The base case with $n = 1$ is trivial. Now suppose that the hypothesis is true for trees on less than $n$ vertices, and we are given a tree $T$ with $n$ vertices and pathwidth $k$, and $r$ is a vertex of $T$ with degree less than $\Delta_0$.

If $r$ is contained in some backbone $B$ of $T$, then let $P = B$. Otherwise, let $P$ be a path from $r$ to an endpoint of a backbone $B$ of $T$. Note that $P$ has at least one edge. As illustrated in Figure 18, draw $P$ horizontally with unit-length edges. Every vertex in $P$ has at most $\Delta_0 - 2$ neighbours in $T \setminus V(P)$, since $r$ has degree less than $\Delta_0$ and the endpoints of a backbone are leaves. At each vertex $x \in P$, the children $\{y_0, y_1, \ldots, y_{\Delta_0 - 3}\}$ of $x$ are positioned below $P$ and on the unit-circle centred at $x$, so that each edge $xy_j$ has slope $\frac{\pi}{2}(1 + j/(\Delta_0 - 2)) \in S$.

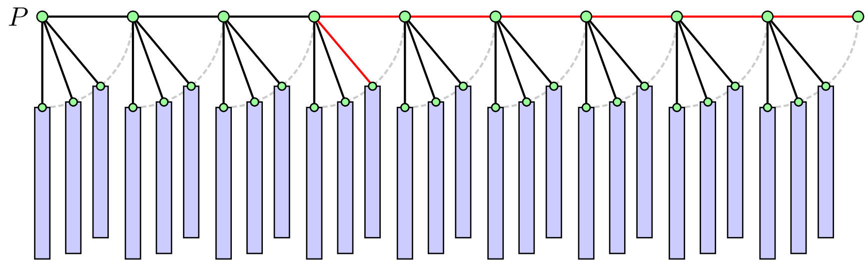


Figure 18: Drawing of $T$ with few slopes and few edge lengths.

Every connected component $T'$ of $T \setminus V(P)$ is a tree rooted at some vertex $r'$ adjacent to a vertex in $P$. By the above layout procedure, $r'$ has already been positioned in the drawing of $T$. If $T'$ is a single vertex, then we no longer need to consider this $T'$.

We consider two types of subtrees $T'$, depending on whether the pathwidth of $T'$ is less than $k$. Suppose that the pathwidth of $T'$ is $k$ (it cannot be more). Then $T' \cap B \neq \emptyset$ since $B$ is a backbone of $T$. Thus $T' \cap B$ is a backbone of $T'$ containing $r'$. Thus we can apply the stronger induction hypothesis in this case.

Every $T'$ has less vertices than $T$, and every $r'$ has degree less than $\Delta_0$ in $T'$. Thus by induction, every $T'$ has a drawing with $r'$ at the top, and every edge of $T'$ has slope in $S$. Furthermore, if the pathwidth of $T'$ is less than $k$, then every edge of $T'$ has length in $\{\ell^0, \ell^1, \ldots, \ell^{2k-3}\}$. Otherwise $r'$ is in a backbone of $T'$, and every edge of $T'$ has length in $\{\ell^0, \ell^1, \ldots, \ell^{2k-2}\}$.

There exists a scale factor $\ell < 1$, depending only on $n_0$ and $\Delta_0$, so that by scaling the drawings of every $T'$ by $\ell$, the widths of the drawings are small enough so that there is no crossings when the drawings are positioned with each $r'$ at its already chosen location. (Note that $\ell$ is the same value at every level of the induction.) Scaling preserves the slopes of the edges. An edge in any $T'$ that had length $\ell^i$ before scaling, now has length $\ell^{i+1}$.

Case 1. $r$ is contained in some backbone $B$ of $T$: By construction, $P = B$. So every $T'$ has pathwidth at most $k - 1$, and thus every edge of $T'$ has length in $\{\ell^1, \ell^2, \ldots, \ell^{2k-2}\}$. All the other edges of $T$ have unit-length. Thus we have a plane drawing of $T$ with edge lengths $\{\ell^0, \ell^1, \ldots, \ell^{2k-2}\}$, as claimed.

Case 2. $r$ is not contained in any backbone of $T$: Every edge in every $T'$ has length in $\{\ell^1, \ell^2, \ldots, \ell^{2k-1}\}$. All the other edges of $T$ have unit-length. Thus we have a plane drawing of $T$ with edge lengths $\{\ell^0, \ell^1, \ldots, \ell^{2k-1}\}$, as claimed. □

**Theorem 8.** *Let $G$ be a graph with $n$ vertices, maximum degree $\Delta$, and treewidth $k$. Then $G$ has a drawing with $\mathcal{O}(k^3 \Delta^4 \log n)$ slopes.*

*Proof.* Ding and Oporowski [11] proved that for some tree $T$, $G$ has a $T$-partition of width at most $\max\{24k\Delta, 1\}$. Let $w = \max\{24k\Delta, 1\}$. For each node $x \in V(T)$, there are at most $w\Delta$ edges of $G$ incident to vertices mapped to $x$. Hence we can assume that $T$ is a forest with maximum degree at most $w\Delta$, as otherwise there is an edge of $T$ with no edge of $G$ mapped to it, in which case the edge of $T$ can be deleted. Similarly, $T$ has at most

$n$ vertices. Scheffler [45] proved that $T$ has pathwidth at most $\log(2n+1)$; see [3]. By Lemma 17, $T$ has a drawing with at most $w\Delta-1$ slopes and at most $2\log(2n+1)-1$ distinct edge lengths. By Lemma 15, $G$ has a drawing in which the number of slopes is at most $w(w\Delta-1)(2\log(2n+1)-1)(w-1)+(w\Delta-1)+w \in \mathcal{O}(w^3\Delta\log n) \subseteq \mathcal{O}(k^3\Delta^4\log n)$. □

**Corollary 6.** *Every $n$-vertex graph with bounded degree and bounded treewidth has a drawing with $\mathcal{O}(\log n)$ slopes.* □

# 7 1-Bend Drawings

While it is an open problem whether every graph has a drawing in which the number of slopes is bounded by the maximum degree, there is a simple solution to this problem if we allow bends. A *1-bend drawing* of a graph $G$ is a drawing of the subdivision of $G$ with one subdivision vertex per edge.

**Theorem 9.** *Every graph $G$ has a 1-bend drawing with $\Delta(G)+1$ slopes.*

*Proof.* Let $S$ be a set of $\Delta(G)+1$ distinct slopes. Suppose the vertices of $G$ have been positioned in the plane. For each vertex $v$ of $G$ and each slope $\ell \in S$, we consider there to be a *slope line* through $v$ with slope $\ell$. Position the vertices of $G$ at distinct points in the plane so that: (1) each slope line intersects exactly one vertex, and (2) no three slope lines intersect at a single point, unless all three are the slope lines of a single vertex. This can be achieved by positioning each vertex in turn, since at each step, there are finitely many forbidden positions.

Consider each slope line to be initially *unused*. Each edge is drawn with one bend, using one slope line at each of its endpoints, in which case, we say these slope lines become *used*. Now draw each edge $vw$ of $G$ in turn. At most $\deg(v)-1$ slope lines at $v$ are used, and at most $\deg(w)-1$ slope lines at $w$ are used. Since $|S| \geq \deg(v)+1$ and $|S| \geq \deg(w)+1$, there are two unused slope lines at $v$, and two unused slope lines at $w$. Thus there is an unused slope line at $v$ that intersects an unused slope line at $w$. Position the bend for $vw$ at this intersection point.

We now prove that this defines a drawing of $G'$. Suppose on the contrary that there is an edge $vu$ of $G'$ and a vertex $w$ of $G'$ that intersects $vu$, and $v \neq w \neq u$. Without loss of generality, $v$ is a vertex of $G$ and $u$ is a subdivision vertex. Since each slope line intersects exactly one vertex of $G$, $w$ is a subdivision vertex of some edge $w_1w_2$ of $G$. Since edges are only drawn on unused slope lines, $w_1 \neq v$ and $w_2 \neq v$. Therefore, the three slope lines containing the edges $w_1w$, $w_2w$ and $vu$ intersect in one point, and all three do not belong to the same vertex. This is a desired contradiction. □

# 8 Open Problems

Here are a few of the numerous open problems related to this research.

**Open Problem 7.** Is there a polynomial time algorithm to test if a graph has a drawing in which the number of segments equals half the number of odd degree vertices?

**Open Problem 8.** Given an edge-colouring of a graph $G$, what is the complexity of determining whether $G$ has a drawing in which monochromatic edges have the same slope?

**Open Problem 9.** What is the minimum number of lines that cover a drawing of a given graph $G$? Obviously, the minimum number of slopes in a drawing of $G$ is at most the minimum number of lines that cover a drawing of $G$, which is at most the minimum number of segments in a drawing of $G$.

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## References

[1] J. Bhasker and S. Sahni, A linear time algorithm to check for the existence of a rectangular dual of a planar triangulated graph. *Networks*, **17(3)**:307–317, 1987.

[2] J. Bhasker and S. Sahni, A linear algorithm to find a rectangular dual of a planar triangulated graph. *Algorithmica*, **3(2)**:247–278, 1988.

[3] H. L. Bodlaender, A partial $k$-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, **209(1-2)**:1–45, 1998.

[4] H. L. Bodlaender and J. Engelfriet, Domino treewidth. *J. Algorithms*, **24(1)**:94–123, 1997.

[5] P. Bose, F. Gómez, P. A. Ramos, and G. T. Toussaintt, Drawing nice projections of objects in space. In F. J. Brandenburg, ed., *Proc. International Symp. on Graph Drawing (GD '95)*, vol. 1027 of *Lecture Notes in Comput. Sci.*, pp. 52–63, Springer, 1996.

[6] J. Czyzowicz, Lattice diagrams with few slopes. *J. Combin. Theory Ser. A*, **56(1)**:96–108, 1991.

[7] J. Czyzowicz, A. Pelc, and I. Rival, Drawing orders with few slopes. *Discrete Math.*, **82(3)**:233–250, 1990.

[8] J. Czyzowicz, A. Pelc, I. Rival, and J. Urrutia, Crooked diagrams with few slopes. *Order*, **7(2)**:133–143, 1990.

[9] H. de Fraysseix, J. Pach, and R. Pollack, How to draw a planar graph on a grid. *Combinatorica*, **10(1)**:41–51, 1990.

[10] M. B. Dillencourt, D. Eppstein, and D. S. Hirschberg, Geometric thickness of complete graphs. *J. Graph Algorithms Appl.*, **4(3)**:5–17, 2000.

[11] G. Ding and B. Oporowski, Some results on tree decomposition of graphs. *J. Graph Theory*, **20(4)**:481–499, 1995.

[12] G. Ding and B. Oporowski, On tree-partitions of graphs. *Discrete Math.*, **149(1-3)**:45–58, 1996.

[13] G. A. Dirac, On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg*, **25**:71–76, 1961.

[14] V. Dujmović and D. R. Wood, Tree-partitions of $k$-trees with applications in graph layout. In H. Bodlaender, ed., *Proc. 29th Workshop on Graph Theoretic Concepts in Computer Science (WG'03)*, vol. 2880 of *Lecture Notes in Comput. Sci.*, pp. 205–217, Springer, 2003.

[15] V. Dujmović and D. R. Wood, On linear layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, to appear.

[16] C. A. Duncan, D. Eppstein, and S. G. Kobourov, The geometric thickness of low degree graphs. In *Proc. 20th ACM Symp. on Computational Geometry (SoCG '04)*, ACM Press, to appear.

[17] P. Eades, M. E. Houle, and R. Webber, Finding the best viewpoints for three-dimensional graph drawings. In G. Di Battista, ed., *Proc. 5th International Symp. on Graph Drawing (GD '97)*, vol. 1353 of *Lecture Notes in Comput. Sci.*, pp. 87–98, Springer, 1997.

[18] D. Eppstein, Dilation-free planar graphs, 1997, `http://www.ics.uci.edu/~eppstein/junkyard/dilation-free/`.

[19] D. Eppstein, Separating thickness from geometric thickness. In J. Pach, ed., *Towards a Theory of Geometric Graphs*, vol. 342 of *Contemporary Mathematics*, pp. 75–86, Amer. Math. Soc., 2004.

[20] P. Erdős, R. Graham, I. Z. Ruzsa, and H. Taylor, Bounds for arrays of dots with distinct slopes on lengths. *Combinatorica*, **12(1)**:39–44, 1992.

[21] I. Fáry, On straight line representation of planar graphs. *Acta Univ. Szeged. Sect. Sci. Math.*, **11**:229–233, 1948.

[22] F. V. Fomin and P. A. Golovach, Interval degree and bandwidth of a graph. *Discrete Appl. Math.*, **129(2-3)**:345–359, 2003.

[23] R. Freese, Automated lattice drawing. In P. W. Eklund, ed., *Concept Lattices. Proc. 2nd International Conf. on Formal Concept Analysis (ICFCA '04)*, vol. 2961 of *Lecture Notes in Comput. Sci.*, pp. 112–127, Springer, 2004.

[24] A. Garg and R. Tamassia, On the computational complexity of upward and rectilinear planarity testing. *SIAM J. Comput.*, **31(2)**:601–625, 2001.

[25] R. Halin, Tree-partitions of infinite graphs. *Discrete Math.*, **97**:203–217, 1991.

[26] M. E. Houle and R. Webber, Approximation algorithms for finding best viewpoints. In S. Whitesides, ed., *Proc. 6th International Symp. on Graph Drawing (GD '98)*, vol. 1547 of *Lecture Notes in Comput. Sci.*, pp. 210–223, Springer, 1998.

[27] R. E. Jamison, Planar configurations which determine few slopes. *Geom. Dedicata*, **16(1)**:17–34, 1984.

[28] R. E. Jamison, Structure of slope-critical configurations. *Geom. Dedicata*, **16(3)**:249–277, 1984.

[29] R. E. Jamison, Few slopes without collinearity. *Discrete Math.*, **60**:199–206, 1986.

[30] R. E. Jamison, Direction trees. *Discrete Comput. Geom.*, **2(3)**:249–254, 1987.

[31] G. Kant, Drawing planar graphs using the canonical ordering. *Algorithmica*, **16(1)**:4–32, 1996.

[32] S. M. Malitz, Graphs with $E$ edges have pagenumber $O(\sqrt{E})$. *J. Algorithms*, **17(1)**:71–84, 1994.

[33] J. L. Martin, Geometry of graph varieties. *Trans. Amer. Math. Soc.*, **355(10)**:4151–4169, 2003.

[34] J. L. Martin, The slopes determined by $n$ points in the plane, 2004, arXiv.org:math.AG/0302106.

[35] J. L. Martin, On the topology of graph picture spaces. *Adv. Math.*, to appear, arXiv.org:math.CO/0307405.

[36] S. Onn and R. Pinchasi, A note on the minimum number of edge-directions of a convex polytope. *Journal of Combinatorial Theory Series A*, to appear.

[37] J. Pach, R. Pinchasi, and M. Sharir, A tight bound for the number of different directions in three dimensions. In *Proc. 19th ACM Symp. on Computational Geometry (SoCG '03)*, pp. 106–113, ACM Press, 2003.

[38] J. Pach, R. Pinchasi, and M. Sharir, Solution of Scott's problem on the number of directions determined by a point set in 3-space. In *Proc. 20th ACM Symp. on Computational Geometry (SoCG '04)*, ACM Press, to appear.

[39] R. E. Peile and H. Taylor, Sets of points with pairwise distinct slopes. *Comput. Math. Appl.*, **39(11)**:109–115, 2000.

[40] M. S. Rahman, S. Nakano, and T. Nishizeki, Rectangular grid drawings of plane graphs. *Comput. Geom.*, **10(3)**:203–220, 1998.

[41] M. S. Rahman, S. Nakano, and T. Nishizeki, A linear algorithm for bend-optimal orthogonal drawings of triconnected cubic plane graphs. *J. Graph Algorithms Appl.*, **3**:31–62, 1999.

[42] M. S. Rahman, S. Nakano, and T. Nishizeki, Rectangular drawings of plane graphs without designated corners. *Comput. Geom.*, **21(3)**:121–138, 2002.

[43] M. S. Rahman, S.-i. Nakano, and T. Nishizeki, Box-rectangular drawings of plane graphs. *J. Algorithms*, **37(2)**:363–398, 2000.

[44] M. S. Rahman, T. Nishizeki, and S. Ghosh, Rectangular drawings of planar graphs. *J. Algorithms*, **50**:62–78, 2004.

[45] P. Scheffler, *Die Baumweite von Graphen als ein Maß für die Kompliziertheit algorithmischer Probleme*. Ph.D. thesis, Akademie der Wissenschaften der DDR, Berlin, Germany, 1989.

[46] P. R. Scott, On the sets of directions determined by $n$ points. *Amer. Math. Monthly*, **77**:502–505, 1970.

[47] D. Seese, Tree-partite graphs and the complexity of algorithms. In L. Budach, ed., *Proc. International Conf. on Fundamentals of Computation Theory*, vol. 199 of *Lecture Notes in Comput. Sci.*, pp. 412–421, Springer, 1985.

[48] M. Suderman, Pathwidth and layered drawings of trees. *Internat. J. Comput. Geom. Appl.*, to appear.

[49] C. Thomassen, Plane representations of graphs. In *Progress in graph theory*, pp. 43–69, Academic Press, Toronto, 1984.

[50] P. Ungar, On diagrams representing maps. *J. London Math. Soc.*, **28**:336–342, 1953.

[51] P. Ungar, $2N$ noncollinear points determine at least $2N$ directions. *J. Combin. Theory Ser. A*, **33(3)**:343–347, 1982.

[52] G. A. Wade and J.-H. Chu, Drawability of complete graphs using a minimal slope set. *The Computer Journal*, **37(2)**:139–142, 1994.

[53] K. Wagner, Bemerkung zum Vierfarbenproblem. *Jber. Deutsch. Math.-Verein.*, **46**:26–32, 1936.

[54] D. R. Wood, Characterisations of intersection graphs by vertex orderings, 2004, submitted. arXiv.org:cs.DM/0404031.

[55] Z. Zhang, A note on arrays of dots with distinct slopes. *Combinatorica*, **13(1)**:127–128, 1993.