NOTES ON LARGE ANGLE CROSSING GRAPHS

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Abstract. A graph G is an α angle crossing (αAC) graph if every pair of crossing edges in G intersect at an angle of at least α. The concept of right angle crossing (RAC) graphs ($\alpha = \pi/2$) was recently introduced by Didimo *et al.* [7]. It was shown that any RAC graph with n vertices has at most $4n-10$ edges and that there are infinitely many values of n for which there exists a RAC graph with n vertices and $4n-10$ edges. In this paper, we give upper and lower bounds for the number of edges in αAC graphs for all $0 < \alpha < \pi/2$.

1 Introduction

The problem of making good graph drawings of relational data sets is a fundamental problem and has been studied extensively, see the books [5, 12, 14, 16]. One measure of a graph drawing algorithm's quality is the number of edge crossings it draws [8, 13, 14, 15]. While some graphs cannot be drawn without edge crossings, some graphs can. These are called planar graphs. According to this metric, "good" algorithms draw graphs with as few edge crossings as possible. This intuition has some scientific validity: experiments by Purchase et al. [17, 18, 20] have shown that performance of humans in path tracing tasks is negatively correlated to the number of edge crossings and to the number of bends in the drawing.

However, recently Huang *et al.* [9, 10, 11] showed, through eye-tracking experiments, that crossings that occur at angles of greater than 70° have very little effect on humans' abilities to interpret graphs. Therefore, graph drawings with crossing are not bad, as long as the crossings occur with large angles between them. This motivated Didimo *et al.* [7] to introduce the so-called right angle crossing (RAC) graphs. A graph G is a RAC graph if any two crossing segments are orthogonal with each other.

In this paper we generalize the concept to α angle crossing (αAC) graphs. A graph G is an (αAC) graph if every pair of crossing edges in G intersect at an angle of at least α. Clearly, αAC graphs are more general than planar graphs and RAC graphs, but how much more so? One measure of generality is the maximum number of edges such a graph can represent. Euler's Formula implies that a planar graph with n vertices has at most $3n-6$ edges. How many edges can an αAC graph have?

1.1 Previous Work

Didimo *et al.* studied $\pi/2$-angle crossing graphs, called right angle crossing (RAC) graphs, and showed that any RAC graph with n vertices has at most $4n-10$ edges and that there exists infinitely many values of n for which there exists a RAC graph with n vertices and $4n-10$ edges. Recently, Angelini *et* *al.* [4] considered some special cases for drawing RAC graphs, for example, acyclic planar RAC digraphs and upward RAC digraphs. They showed that there exist acyclic planar digraphs not admitting any straight-line upward RAC drawing and that the corresponding decision problem is NP-hard. They also gave a construction of digraphs whose straight-line upward RAC drawings require exponential area.

For $\alpha > \pi/3$, an αAC graph has no three edges that mutually intersect since, otherwise, one of the pairs of edges must intersect at an angle that is at most $\pi/3$. Geometric graphs with no three pairwise crossing edges are known as *quasiplanar graphs* [3]. Ackerman and Tardos [2, Theorem 5] have shown that any quasiplanar graph on n vertices has at most $6.5n - 20$ edges.

For $\alpha > \pi/4$, an αAC graph has no four pairwise crossing edges. Ackerman [1] has shown that any such graph has at most $36n - 72$ edges. It remains an open problem whether, for any $k \geq 5$, a graph with no k-pairwise crossing edges has a linear number of edges. The best known upper bound of $O(n \log n)$ on the number of edges in such a graph is due to Valtr [19, Theorem 3].

1.2 New Results

The current paper gives upper and lower bounds on the number of edges in αAC graphs. In Section 2 we show that, for any $0 < \alpha < \pi/2$, the maximum number of edges in an αAC graph is at most $(\pi/\alpha)(3n - 6)$. In Section 3, we give constructions that essentially match this upper bound when $\alpha = \pi/k - \epsilon$, for $k = 2, 3, 4, 6$ and any $\epsilon > 0$. Finally, in Section 4 we use a charging argument similar to the one used by Ackerman and Tardos [2] to prove that, for $2\pi/5 < \alpha < \pi/2$, the number of edges in an αAC graph is bounded by $6n - 12$.

2 A Uniform Upper Bound

In this section, we give an upper bound of $(\pi/\alpha)(3n - 6)$ on the number of edges in an αAC graph. This upper bound captures the intuition that an αAC graph can be viewed as the union of π/α planar graphs. The only trouble with this intuition is that π/α is not necessarily an integer so we have a problem of determining the number of edges in a fraction of a planar graph.

Theorem 1. *Let G be an αAC graph with n vertices, for some $0 < \alpha < \pi/2$. Then G has at most $(\pi/\alpha)(3n - 6)$ edges.*

Proof. Define the *direction* of an edge xy whose lower endpoint is x (in the case of a horizontal edge, take x as the left endpoint) as the angle $\angle wxy$ where $w = x + (1, 0)$. The direction of an edge xy is therefore a real number in the interval $[0, \pi)$. Now, take a random rotation G' of G and partition the edges of G' into groups $G_1, \ldots, G_r$ where $r = \lceil \pi/\alpha \rceil$, and G_i, $1 \leq i \leq r$, contains all edges of G' whose direction is in the interval $[\alpha(i - 1), \alpha i)$.

Note that no two edges of G_i cross each other, so each G_i is a planar graph that, by Euler's Formula, has at most $3n - 6$ edges. Furthermore, since G' is a random rotation, the expected number of edges in G_r is $(\pi \bmod \alpha)|E(G)|$. In particular, there must exist some rotation G' of G such that $|E(G')| \leq (\pi \bmod \alpha)|E(G)|$. Therefore,

$$E(G) \leq \lfloor \pi/\alpha \rfloor (3n - 6) + (\pi \bmod \alpha)|E(G)| \ . \qquad (1)$$

Rearranging (1) yields

$$|E(G)| \leq \frac{\lfloor \pi/\alpha \rfloor (3n-6)}{1 - \pi \bmod \alpha} = (\pi/\alpha)(3n-6) \ ,$$

as required. □

3 Lower Bounds

Theorem 2. *For any $\epsilon > 0$, there exist $(\pi/2 - \epsilon)AC$ graphs that have n vertices and $6n - o(n)$ edges.*

Proof. Let $m = n^{1/3}$ be a positive integer and assume for simplicity that $\sqrt{n/m}$ is an integer. The construction is based on the $\sqrt{n/m} \times \sqrt{n/m}$ square grid and is illustrated in Figure 1(a)–(b). At each grid point (i,j) place a *group*, denoted $S_{i,j}$, of m vertices. The diameter of a group is at most ϵ/c, where c is a large constant.

Let $(0,0)$ and $(\sqrt{n/m}, \sqrt{n/m})$ be the bottom left grid point and the top right grid point, respectively. Consider a group $S_{i,j}$ of m vertices at grid point (i,j). If $(i+j) \bmod 2 = 0$ then the vertices, $s^1_{i,j}, \ldots, s^m_{i,j}$, in $S_{i,j}$ lie on a line with slope -1 with $s^1_{i,j}$ at the grid point (i,j) and the remaining points evenly spread out with an inter point distance of $\epsilon/c(m-1)$. Otherwise, if $(i+j) \bmod 2 = 1$ then the vertices, $s^1_{i,j}, \ldots, s^m_{i,j}$, in $S_{i,j}$ lie on a line with slope -1 with $s^1_{i,j}$ at $(i + \epsilon/2c, j - \epsilon/2c)$ and the remaining points evenly spread out with an inter point distance of $\epsilon/c(m-1)$, as shown in Figure 1(b).

A vertex $s^k_{i,j}$ in $S_{i,j}$ is connected to the vertices $s^k_{i-2,j}, s^k_{i+2,j}, s^k_{i,j-2}$ and $s^k_{i-2,j+2}$, if they exist. It is also connected to the vertices $s^{k-1}_{i',j'}$ and $s^{k+1}_{i',j'}$, for $i' = i-1$, $i' = i+1$, $j' = j-1$ and $j' = j+1$, if they exist. It is easy to verify that choosing c large enough will guarantee that the angle between any pair of crossing edges is at least $\pi/2 - \epsilon$.

It remains to count the number of edges. Note that the degree of every vertex $s^k_{i,j}$ is 12 if $1 < k < m$ and $3 \leq i,j \leq \sqrt{n/m} - 2$. The total number of such vertices is:

$$(\sqrt{n/m} - 4)^2 \cdot (m-2) = n - O(n/m + \sqrt{nm}).$$

Since $m = n^{1/3}$ the total degree is $12n - o(n)$ which immediately gives the bound stated in the theorem. □

Theorem 3. *For any $\epsilon > 0$, there exist $(\pi/3 - \epsilon)AC$ graphs that have n vertices and $9n - o(n)$ edges.*

Proof. The construction is based on the hexagonal lattice as illustrated in Figure 2. The proof is similar to the the proof of Theorem 2. □

Theorem 4. *For any $\epsilon > 0$, there exist $(\pi/4 - \epsilon)AC$ graphs that have n vertices and $12n - o(n)$ edges.*

Proof. The construction is based on the square lattice as illustrated in Figure 1(c). The proof is similar to the the proof of Theorem 2. □

Theorem 5. *For any $\epsilon > 0$, there exist $(\pi/6 - \epsilon)AC$ graphs that have n vertices and $18n - o(n)$ edges.*

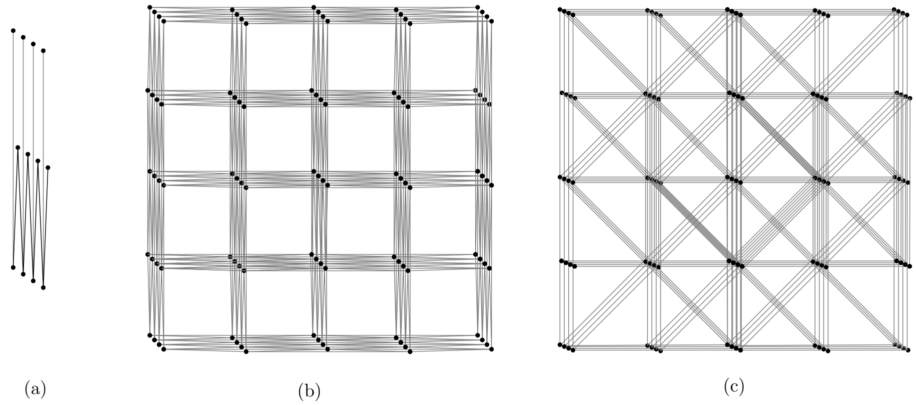

Figure 1: The lower bound constructions in the proofs of Theorem 2 and Theorem 4.

Proof. The construction is based on the hexagonal lattice as illustrated in Figure 2. The proof is similar to the the proof of Theorem 2. □

Open Problem Can the lower bounds in this section be generalized to a general bound?

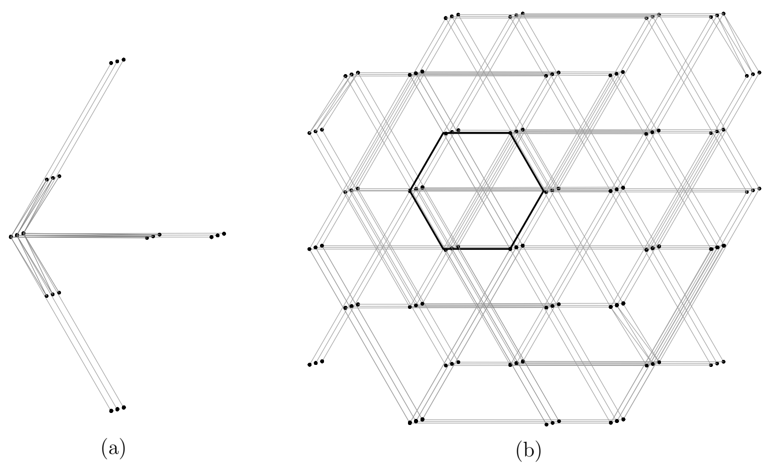

Figure 2: The lower bound construction in the proof of Theorem 3.

4 Charging Arguments

In this section we derive upper bounds using charging arguments similar to those used by Ackerman and Tardos [2] and Ackerman [1]. Let G be an αAC graph. We denote by G' the planar graph obtained by introducing a vertex at each point in which a pair of edges in G crosses (thereby subdividing) two edges of G.

For a face f of G', we denote by $|f|$ the number of steps taken while traversing the boundary of f in counterclockwise order so that, if we walk along an edge twice during the traversal, then it contributes twice to $|f|$. Let $v(f)$ denote the number of steps of this traversal during which a vertex of G (as opposed to a vertex introduced in G') is encountered. For each face f of G' define the *initial charge* of f as

$$\mathrm{ch}(f) = |f| + v(f) - 4 \ .$$

Ackerman and Tardos show, using two applications of Euler's formula, that

$$\sum_{f \in G'} \mathrm{ch}(f) = 4n - 8 \ .$$

We call a face f of G' a *k-shape* if $v(f) = k$ and f is a *shape*. For example, a 2-pentagon is a face of G' with $|f| = 5$ and $v(f) = 2$.

As a warm-up, and introduction to charging arguments, we offer an alternate proof to the upper bound presented by Didimo *et al.* in [7].

Theorem 6. *A RAC graph with $n \geq 4$ vertices has at most $4n - 10$ edges.*

Proof. Let G be a maximal RAC graph on n vertices, and define G' and ch as above. We claim that, for every face f of G', $\mathrm{ch}(f) \geq v(G)/2$. To see this, observe that the claim is certainly true if $|f| \geq 4$. On the other hand, if $|f| = 3$ then, by the RAC property, $v(f) \geq 2$, so it is also true in this case. Therefore,

$$4n - 8 = \sum_{f \in G'} \mathrm{ch}(f) \geq \sum_{f \in G'} v(f)/2 = \sum_{v \in G} \deg(v)/2 = |E(G)| \ ,$$

which proves that $E(G) \leq 4n - 8$.

To improve the above bound, observe that, since G is maximal all vertices on the outer face, f, of G' are vertices of G. If $|f| \geq 4$ then $\mathrm{ch}(f) \geq v(f)/2 + 2$, so in this case, proceeding as above, we have

$$4n - 8 - 2 \geq |E(G)|$$

and we are done. Otherwise, the outer face of G' is a 3-triangle and $\mathrm{ch}(f) = v(f)/2 + 1/2$. Consider the internal faces of G' incident to the three edges of f. Because G is maximal, and $n \geq 4$, there must be three such faces and each of these three faces, f', has $v(f') \geq 2$. Furthermore, at most one of these faces is a 2-triangle.[1] A straightforward case analysis shows that the other two faces must have $\mathrm{ch}(f') \geq v(f')/2 + 1/2$, with equality if and only if f' is a 3-triangle. Therefore, we have

$$4n - 8 - 3/2 \geq |E(G)|$$

which, implies that $|E(G)| \leq 4n - 10$ since $|E(G)|$ is an integer. □

Next, we prove an upper bound for $\alpha > 2\pi/5$ that improves on the $6.5n - 20$ upper bound that follows from Ackerman and Tardos' bound on quasiplanar graphs.

Theorem 7. *Let G be an αAC graph with n vertices, for $\alpha > 2\pi/5$. Then G has at most $6n - 12$ edges.*

[1] This is proven by a simple geometric argument that shows for any triangle f, two right-angle triangles that are interior to f and each share an edge with f must overlap.

Proof. We will redistribute the charge in the graph G' to obtain a new charge ch′ such that $\mathrm{ch}'(f) \geq v(f)/3$ for every face f of G. In this way, we get

$$4n - 8 = \sum_{f \in G'} \mathrm{ch}(f) = \sum_{f \in G'} \mathrm{ch}'(f) \geq \sum_{f \in G'} v(f)/3 = \sum_{v \in G} \deg(v)/3 = 2|E(G)|/3 \ ,$$

which we rewrite to get $|E(G)| \leq 6n - 12$.

The charge $\mathrm{ch}'(f)$ is obtained as follows. Let f be any 1-triangle of G'. (Note that $\mathrm{ch}(f) = 0$.) That is, f is a triangle formed by two edges e_1 and e_2 that meet at a vertex x of G and an edge e that crosses e_1 and e_2. Imagine walking along the bisector of e_1 and e_2 (starting in the interior of f) until reaching a face f' such that f' is not a 0-quadrilateral. To see why such an f' exists, observe that if we encounter nothing but 0-quadrilaterals we will eventually reach a face that contains an endpoint of e_1 or e_2 and is therefore not a 0-quadrilateral.

Adjust the charges at f and f' by subtracting 1/3 from $\mathrm{ch}(f')$ and adding 1/3 to $\mathrm{ch}(f)$. It is helpful to think of the charge as leaving f' through the last edge e' traversed in the walk. Note that neither endpoint of e' is a vertex of G. This implies that for a face f', the amount of charge that leaves f' is at most

$$\ell(f') \leq \begin{cases} |f'| & \text{if } v(f') = 0 \\ |f'| - v(f') - 1 & \text{otherwise.} \end{cases} \tag{2}$$

Let ch′ be the charge obtained after performing this redistribution of charge for every 1-triangle f. We claim that $\mathrm{ch}'(f) \geq v(f)/3$. To see this, we need only run through a few cases that can be verified using (2) and the following observations:

1. If $|f| \geq 6$, then $\ell(f) \leq |f|/3$, so $\mathrm{ch}'(f) \geq v(f) \geq v(f)/3$.
2. If $|f| = 5$, then $v(f) \geq 1$ since, otherwise, f has two edges on its boundary that cross at an angle of less than or equal to $2\pi/5$.
3. If $|f| = 4$, and f is a 0-quadrilateral then $\ell(f) = 0$, by construction.
4. If $|f| = 3$, and f is a 1-triangle then $\mathrm{ch}'(f) = 1/3$, by construction.
5. If $|f| = 3$ then $v(f) \geq 1$ since, otherwise, f has two edges on its boundary that cross at an angle less of at most $\pi/3 < 2\pi/5$.

This completes the proof. □

5 Notes

Theorem 7 appears to be true even for $\alpha > \pi/3$, but we have not been able to prove it. The problem occurs because 0-pentagons can finish with a charge of $-2/3$ or $-1/3$. (See Figure 3.)

A proof could maybe look for extra charge near the vertices of the penta*gram* that created this pentagon, but it is easy to make gadgets so that the faces surrounding those vertices have no extra

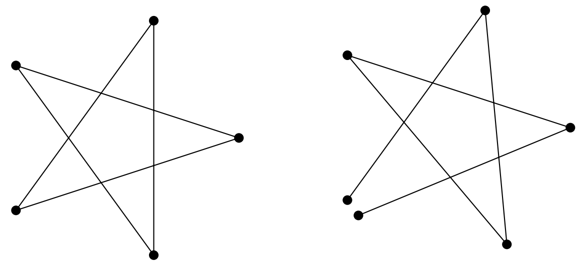

Figure 3: Pentagrams lead to 0 pentagons with negative charge.

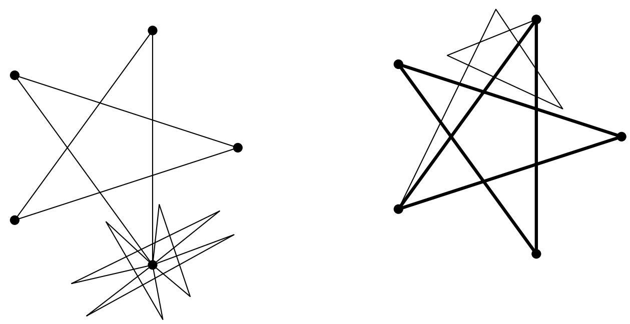

Figure 4: Pentagrams can have 0 extra charge at their vertices and 0 extra charge at the vertices of a pentagon.

charge. Another option is to look for extra charge near the vertices of the penta*gon* itself. Again, it is not too hard to to make them have no extra charge. (See Figure 4.)

We also tried to follow the Ackerman-Tardos proof more closely. Namely, we distribute the charge so that $\mathrm{ch}'(f) \geq v(f)/5$ and then prove that there is leftover charge at the faces around each vertex. For this to give a bound of $6n$ we would need the extra charge at each vertex to be $8/5$. Unfortunately, the limiting case in Ackerman-Tardos is $7/5$ and this is realizable even with crossing angles arbitrarily close to $\pi/2$. (See Figure 5.)

Finally, we can take a more global approach. Discharging rules define a directed graph among the faces (and possibly vertices) of G'. An edge ab indicates that a charge of x travels from a to b, for

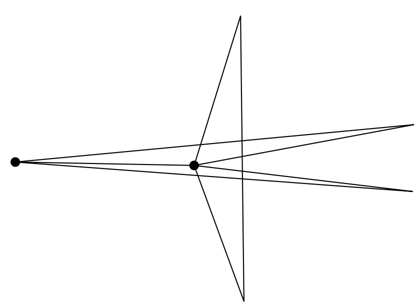

Figure 5: The Ackerman-Tardos proof cannot even prove a bound of $6n$ for crossing angles of $\pi/2 - \epsilon$.

some number x ($x = 1/3$ in our argument). The graph has to respect some flow rules. For example, in Theorem 7 we have

$$\text{outdeg}(a) - \text{indeg}(a) \leq 3(|f| + 2v(f)/3 - 4) \ ,$$

where indeg and outdeg denote the in and out degree. The goal would be to define discharging paths recursively and then show that the recursion terminates (i.e. that the resulting graph is acyclic) and that the flow rule is satisfied.

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