Improved Upper Bounds on the Crossing Number

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Abstract

The crossing number of a graph is the minimum number of crossings in a drawing of the graph in the plane. Our main result is that every graph G that excludes a fixed graph as a minor has crossing number $\mathcal{O}(\Delta n)$, where G has n vertices and maximum degree Δ. This dependence on Δ and n is best possible. This result answers an open question of Wood and Telle [New York J. Mathematics, 2007], who proved the previous best known bound of $\mathcal{O}(\Delta^2 n)$.

In addition, we prove that every K_5-minor-free graph G has crossing number at most $\sum_{v\in V(G)} \deg(v)^2$, which again is the best possible dependence on the degrees of G. Finally, we also study the convex and rectilinear crossing numbers and prove a $\mathcal{O}(\Delta n)$ bound for the convex crossing number of bounded pathwidth graphs, and a $\sum_{v\in V(G)} \deg(v)^2$ bound for the rectilinear crossing number of $K_{3,3}$-minor-free graphs.

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1 Introduction

The *crossing number* of a graph[1] G, denoted by $\mathsf{cr}(G)$, is the minimum number of crossings in a drawing[2] of G in the plane; see [18, 35, 53] for surveys. The crossing number is an important measure of the non-planarity of a graph [52], with applications in discrete and computational geometry [34, 51] and VLSI circuit design [4, 28, 29]. In information visualisation, one of the most important measures of the quality of a graph drawing is the number of crossings [38–40].

Computing the crossing number is $\mathcal{NP}$-hard [20], and remains so for simple cubic graphs [24, 37]. Moreover, the exact or even asymptotic crossing number is not known for specific graph families, such as complete graphs [44], complete bipartite graphs [31, 42, 44], and cartesian products [2, 6, 22, 43]. On the other hand, for fixed k, Kawarabayashi and Reed [27] developed a linear-time algorithm that decides whether a given graph has crossing number at most k, and if this is the case, computes a drawing of the graph with at most k crossings.

Given that the crossing number seems so difficult, it is natural to focus on asymptotic bounds rather than exact values. The 'crossing lemma', conjectured by Erdős and Guy [18] and first proved by Leighton [28] and Ajtai et al. [3], gives such a lower bound. It states that every graph G with average degree greater than $6+\epsilon$ has $\mathsf{cr}(G) \geq c_\epsilon \frac{|E(G)|^3}{|V(G)|^2}$. Other general lower bound techniques that arose out of the work of Leighton [28, 29] include the bisection/cutwidth method [16, 33, 49, 50] and the embedding method [48, 49].

Upper bounds on the crossing number of general families of graphs have been less studied, and are the focus of this paper. Obviously $\mathsf{cr}(G) \leq \binom{|E(G)|}{2}$ for every graph G. A family of graphs has *linear* crossing number if $\mathsf{cr}(G) \leq c\,|VG|$ for every graph G in the family, for some constant c. The following theorem of Pach and Tóth [36] established that graphs of bounded genus[3] and bounded degree have linear crossing number.

Theorem 1.1 ([36]). *For every integer $\gamma \geq 0$, there are constants c and c', such that every graph G with orientable genus γ has crossing number* $\mathsf{cr}(G) \leq c\sum_{v\in V(G)} \deg(v)^2 \leq c'\Delta(G)\cdot|V(G)|$.

Böröczky et al. [10] extended Theorem 1.1 to graphs of bounded non-orientable genus. Djidjev and Vrt̆o [17] greatly improved the dependence on γ in Theorem 1.1, by proving that $\mathsf{cr}(G) \leq c\gamma\cdot\Delta(G)\cdot|V(G)|$. Wood and Telle [54] proved that bounded-degree graphs that exclude a fixed graph as a minor[4] have linear crossing number.

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[1]We consider graphs G that are undirected, simple, and finite. Let $V(G)$ and $E(G)$ respectively be the vertex and edge sets of G. Let $|V(G)| := |V(G)|$ and $|E(G)| := |E(G)|$. For each vertex v of a graph G, let $N_G(v) := \{w \in V(G) : vw \in E(G)\}$ be the neighbourhood of v in G. The *degree* of v, denoted by $\deg_G(v)$, is $|N_G(v)|$. When the graph is clear from the context, we write $\deg_G(v)$. Let $\Delta(G)$ be the maximum degree of a vertex of G.

[2]A *drawing* of a graph represents each vertex by a distinct point in the plane, and represents each edge by a simple closed curve between its endpoints, such that the only vertices an edge intersects are its own endpoints, and no three edges intersect at a common point (except at a common endpoint). A drawing is *rectilinear* if each edge is a line-segment, and is *convex* if in addition the vertices are in convex position. A *crossing* is a point of intersection between two edges (other than a common endpoint). A drawing with no crossings is *crossing-free*. A graph is *planar* if it has a crossing-free drawing.

[3]Let $\mathbb{S}_\gamma$ be the orientable surface with $\gamma \geq 0$ handles. An *embedding* of a graph in $\mathbb{S}_\gamma$ is a crossing-free drawing in $\mathbb{S}_\gamma$. A 2-*cell embedding* is an embedding in which each region of the surface (bounded by edges of the graph) is an open disk. The (*orientable*) *genus* of a graph G is the minimum γ such that G has a 2-cell embedding in $\mathbb{S}_\gamma$. In what follows, by a *face* we mean the set of vertices on the boundary of the face. Let $F(G)$ be the set of faces in an embedded graph G. See the monograph by Mohar and Thomassen [30] for a thorough treatment of graphs on surfaces.

[4]Let vw be an edge of a graph G. Let G' be the graph obtained by identifying the vertices v and w, deleting loops, and replacing parallel edges by a single edge. Then G' is obtained from G by *contracting* vw. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. A family of graphs $\mathcal{F}$ is *minor-closed*

Theorem 1.2 ([54])**.** *For every graph H, there is a constant $c = c(H)$, such that every H-minor-free graph G has crossing number* $\mathsf{cr}(G) \leq c\,\Delta(G)^2 \cdot |V(G)|$.

Theorem 1.2 is stronger than Theorem 1.1 in the sense that graphs of bounded genus exclude a fixed graph as a minor, but there are graphs with a fixed excluded minor and unbounded genus. On the other hand, Theorem 1.1 has smaller dependence on Δ than Theorem 1.2. For other recent work on minors and crossing number see [7–9, 19, 21, 23, 24, 32, 37].

Note that, for any reasonably general class of graphs to have linear crossing number, excluding a fixed minor and bounding the maximum degree (as in Theorem 1.2) is unavoidable. For example, $K_{3,n}$ has no K_5-minor, yet has $\Omega(n^2)$ crossing number [31, 42]. Conversely, bounded degree does not by itself guarantee linear crossing number. For example, a random cubic graph on n vertices has $\Omega(n)$ bisection width [11, 13], which implies that it has $\Omega(n^2)$ crossing number [16, 28].

Pach and Tóth [36] proved that the upper bound in Theorem 1.1 is best possible, in the sense that for all Δ and n, there is a toroidal graph with n vertices and maximum degree Δ whose crossing number is $\Omega(\Delta n)$. This graph has no K_8-minor, but has a K_7-minor. In Section 2 we extend this $\Omega(\Delta n)$ lower bound to graphs with no $K_{3,3}$-minor, no K_5-minor, and more generally, no K_h-minor. Our main result is to prove a matching upper bound for all graphs excluding a fixed minor. That is, we improve the quadratic dependence on $\Delta(G)$ in Theorem 1.2 to linear.

Theorem 1.3. *For every graph H there is a constant $c = c(H)$, such that every H-minor-free graph G has a crossing number at most $c\Delta(G) \cdot |V(G)|$.*

While our upper bound in Theorem 1.3 is optimal in terms of $\Delta(G)$ and $|V(G)|$, it remains open whether every graph excluding a fixed minor has $\mathcal{O}(\sum_v \deg(v)^2)$ crossing number, as is the case for graphs of bounded genus. Note that a $\sum_v \deg(v)^2$ upper bound is stronger than a $\Delta(G) \cdot |V(G)|$ upper bound. In particular, for every graph G with bounded average degree (such as graphs with bounded genus or those excluding a fixed minor), $\sum_v \deg(v)^2 \leq \Delta(G) \sum_v \deg(v) = 2\Delta(G) \cdot |E(G)| \leq c\Delta(G) \cdot |V(G)|$. Wood and Telle [54] conjectured that every graph excluding a fixed minor has $\mathcal{O}(\sum_v \deg(v)^2)$ crossing number. In Section 3, we establish this conjecture for K_5-minor-free graphs, and prove the same bound on the rectilinear crossing number[5] of $K_{3,3}$-minor-free graphs. In addition to these results, in Section 4, we prove optimal bounds on the convex crossing number of interval graphs, chordal graphs, and bounded pathwidth graphs.

It is worth noting that our proof is constructive, assuming a structural decomposition (Theorem 5.2) by Robertson and Seymour [46] is given. Demaine et al. [12] gave a polynomial-time algorithm to compute this decomposition. Consequently, our proof can be converted into a polynomial-time algorithm that given a graph G excluding a fixed minor, finds a drawing of G with the claimed number of crossings.

2 Lower Bounds

In this section we describe graphs that provide lower bounds on the crossing number. The constructions are variations on those by Pach and Tóth [36]. We include them here to motivate our interest in matching upper bounds in later sections.

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if $G \in \mathcal{F}$ implies that every minor of G is in $\mathcal{F}$. $\mathcal{F}$ is *proper* if it is not the family of all graphs. A deep theorem of Robertson and Seymour [47] states that every proper minor-closed family can be characterised by a finite family of excluded minors. Every proper minor-closed family is a subset of the H-minor-free graphs for some graph H. We thus focus on minor-closed families with one excluded minor.

[5]The *rectilinear crossing number* of a graph G, denoted by $\overline{\mathsf{cr}}(G)$, (respectively, *convex crossing number*, denoted by $\mathsf{cr}^\star(G)$) is the minimum number of crossings in a rectilinear (convex) drawing of G.

Lemma 2.1. *For all positive integers Δ and n, such that $\Delta \equiv 0 \pmod 4$ and $n \equiv 0 \pmod{5(\Delta/2-1)}$, there is a (chordal) $K_{3,3}$-minor-free graph G with $|V(G)| = n$, $\Delta(G) = \Delta$, and*

$$\mathsf{cr}(G) = \frac{\Delta n}{40}\Big(1 + \frac{2}{\Delta - 5}\Big) > \frac{\Delta n}{40}.$$

Proof. Start with K_5 as the base graph. For each edge vw of K_5, add $\Delta/4 - 1$ new vertices, each adjacent to v and w. The resulting graph G' is chordal and $K_{3,3}$-minor-free, $\Delta(G') = \Delta$, and $|V(G')| = 5(\Delta/2 - 1)$. Take $\frac{n}{5(\Delta/2-1)}$ disjoint copies of G' to obtain a $K_{3,3}$-minor-free graph G on n vertices and maximum degree Δ. Thus $\mathsf{cr}(G) = \mathsf{cr}(G')\frac{n}{5(\Delta/2-1)}$. A standard technique proves that $\mathsf{cr}(G') = (\Delta/4)^2$. Thus $\mathsf{cr}(G) = (\Delta/4)^2\frac{n}{5(\Delta/2-1)} = \frac{\Delta n}{40}(1 + \frac{2}{\Delta-2})$, as claimed. □

A similar technique gives the following lemma.

Lemma 2.2. *Given any set of positive integers $D = \{2, d_1, \dots, d_p\}$ such that for each $d_i \in D \setminus \{2\}$, $d_i \equiv 0 \pmod 4$, there are infinitely many (chordal) $K_{3,3}$ minor-free graphs G such that the degree set of G is D and*

$$\mathsf{cr}(G) > \frac{1}{200}\sum_{v \in V(G)} \deg(v)^2.$$

Proof. For each $d_i \in D \setminus \{2\}$, let $n_i = \frac{5}{2}d_i - 5$. By Lemma 2.1, there is a (chordal) $K_{3,3}$-minor-free graph G_i with five vertices of degree d_i and $n_i - 5$ vertices of degree 2, such that

$$\mathsf{cr}(G_i) > \frac{1}{40}d_i n_i > \frac{1}{200}\big(5d_i^2 + (n_i - 5)2^2\big) = \frac{1}{200}\sum_{v \in V(G_i)} \deg(v)^2.$$

Every graph G created by taking one or more disjoint copies of each of $G_1, \dots, G_p$ is $K_{3,3}$-minor-free with degree set D, and $\mathsf{cr}(G) \geq \frac{1}{200}\sum_{v \in V(G)} \deg(v)^2$. □

The above results generalize to K_h-minor-free graphs, for $h \geq 5$.

Lemma 2.3. *For every integer $h \geq 5$ and every Δ such that $\Delta \equiv 0 \pmod{h-2}$ for $h \geq 6$ and $\Delta \equiv 0 \pmod 3$ for $h = 5$, there exists infinitely many K_h-minor-free graphs G with $\Delta(G) = \Delta$ and*

$$\mathsf{cr}(G) \geq ch \cdot \Delta(G) \cdot |V(G)|,$$

for some absolute constant c. Moreover, G is chordal for $h \geq 6$.

Proof Sketch. For $h = 5$, use $K_{3,3}$ as the base graph. For $h \geq 6$, use K_{h-1} as the base graph. The remaining arguments follow the proof of Lemma 2.1 and use the fact that $\mathsf{cr}(K_{3,3}) = 1$ and $\mathsf{cr}(K_{h-1}) \in \Omega(h^4)$. □

3 Drawings Based on Planar Decompositions

Let G and D be graphs, such that each vertex of D is a set of vertices of G (called a *bag*). For each vertex v of G, let $D(v)$ be the subgraph of D induced by the bags that contain v. Then D is a *decomposition* of G if:

- $D(v)$ is connected and nonempty for each vertex v of G, and
- $D(v)$ and $D(w)$ touch[6] for each edge vw of G.

[6]Let A and B be subgraphs of a graph G. Then A and B *intersect* if $V(A) \cap V(B) \neq \emptyset$, and A and B *touch* if they intersect or $v \in V(A)$ and $w \in V(B)$ for some edge vw of G.

Decompositions, when D is a tree, were introduced by Robertson and Seymour [45]. Diestel and Kühn [15] first generalised the definition for arbitrary graphs D.

Let D be a decomposition of a graph G. The *width* of D is the maximum cardinality of a bag. Let v be a vertex of G. The number of bags in D that contain v is the *spread* of v in D. The *spread* of D is the maximum spread of a vertex of G. A decomposition D of G is a *partition* if every vertex of G has spread 1.The *order* of D is the number of bags. D has *linear order* if $|V(D)| \leq c\,|V(G)|$ for some constant c. If the graph D is a tree, then the decomposition D is a *tree decomposition*. If the graph D is a path, then the decomposition D is a *path decomposition*. The decomposition D is *planar* if the graph D is planar.

A decomposition D of a graph G is *strong* if $D(v)$ and $D(w)$ intersect for each edge vw of G. The *tree-width* (*pathwidth*) of G, is 1 less than the minimum width of a strong tree (path) decomposition of G. Tree-width is particularly important in structural and algorithmic graph theory; see the surveys [5, 41].

Wood and Telle [54] showed that planar decompositions were closely related to crossing number. The next result improves a bound in [54] from $(p-1)\,\Delta(G)\,|E(G)|$ to $(p-1)\sum_v \deg(v)^2$.

Lemma 3.1. *Every graph G with a planar partition H of width p has a rectilinear drawing in which each edge crosses at most $2\,\Delta(G)\,(p-1)$ other edges. The total number of crossings,*

$$\overline{\mathsf{cr}}(G) \;\leq\; (p-1) \sum_{v\in V(G)} \deg(v)^2.$$

Proof. The following drawing algorithm is in [54]. By the Fáry-Wagner Theorem, H has a rectilinear drawing with no crossings. Let $\epsilon > 0$. Let $D_\epsilon(B)$ be the disc of radius ϵ centred at each bag B of H. For each edge BC of H, let $D_\epsilon(BC)$ be the union of all line-segments with one endpoint in $D_\epsilon(B)$ and one endpoint in $D_\epsilon(C)$. For some $\epsilon > 0$, we have $D_\epsilon(B)\cap D_\epsilon(C) = \emptyset$ for all distinct bags B and C of H, and $D_\epsilon(BC)\cap D_\epsilon(PQ) = \emptyset$ for all edges BC and PQ of H that have no endpoint in common. For each vertex v of G in bag B of H, position v inside $D_\epsilon(B)$ so that $V(G)$ is in general position (no three collinear). Draw every edge of G straight. Thus no edge passes through a vertex. Suppose that two edges e and f cross. Then e and f have distinct endpoints in a common bag, as otherwise two edges in H would cross. (The analysis that follows is new.) Say v_i is an endpoint of e and v_j is an endpoint of f, where $\{v_1,\dots,v_p\}$ is some bag with $\deg(v_1)\leq\dots\leq\deg(v_p)$. Without loss of generality $i<j$. Charge the crossing to v_j. The number of crossings charged to v_j is at most $\sum_{i<j}\deg(v_i)\cdot\deg(v_j) \leq (j-1)\deg(v_j)^2$. So the total number of crossings is as claimed. □

Wood and Telle [54] proved that every $K_{3,3}$-minor-free graph has a planar partition of width 2. Thus Lemma 3.1 implies that every $K_{3,3}$-minor-free graph G has rectilinear crossing number

$$\overline{\mathsf{cr}}(G) \;\leq\; \sum_{v\in V(G)} \deg(v)^2.$$

Lemma 3.2. *Suppose that D is a planar decomposition of a graph G with width p, in which each vertex v of G has spread at most $s(v)$. Then G has crossing number*

$$\mathsf{cr}(G) \;\leq\; (p-1) \sum_{v\in V(G)} s(v)\cdot\deg(v)^2$$

Moreover, G has a drawing with the claimed number of crossings, in which each edge vw is represented by a polyline with at most $s(v)+s(w)-2$ bends.

Proof. For each vertex v of G, let $X(v)$ be a bag of D that contains v. For each edge vw of G, let $P(vw)$ be a minimum length path in D between $X(v)$ and $X(w)$, such that v or w is in every bag in $P(vw)$. Let G' be the subdivision of G obtained by subdividing each edge vw of G once for each internal bag in $P(vw)$. Consider each division vertex x of vw to *belong* to v if x is in a bag that contains v. If x is in a bag that contains both v and w, then arbitrarily choose v or w to be the owner of x. Observe that D defines a planar partition of G'. Apply Lemma 3.1 to G'. Consider a vertex x of G', where x belongs to some vertex v. The number of crossings charged to x is at most $(p-1)\deg(v)^2$. Thus the number of crossings charged to vertices that belong to v (including v itself) is at most $(p-1)\cdot s(v)\cdot\deg(v)^2$. Hence the total number of crossings is as claimed. □

Lemma 3.3. *Let D be a planar decomposition of a graph G, in which every bag is a clique in G, and two edges appear in at most c common bags. Then*

$$\mathsf{cr}(G) \leq 4c \sum_{vw\in E(G)} \deg(v)\deg(w).$$

Proof. Draw G as in the proof of Lemma 3.2. Consider two edges vw and xy that cross. Each crossing between vw and xy can be charged to a bag B that contains v or w, and x or y. Since B is a clique, each crossing between vw and xy can be charged to an edge vx, vy, wx, or wy in B. For each such edge $pq \in \{vx, vy, wx, wy\}$ and each bag B containing both p and q, at most one crossing between vw and xy is charged to pq in B. Thus at most $4c$ crossings between an edge incident to p and an edge incident to q are charged to pq. Thus the number of crossings charged to pq is at most $4c\deg(p)\deg(q)$. Thus the total number of crossings is as claimed. □

Wood and Telle [54] constructed planar decompositions of K_5-minor-free graphs as follows.

Lemma 3.4 ([54])**.** *Let G be a K_5-minor-free graph. Then G has a set of at most $|V(G)|-2$ edges E such that if V is the set of vertices of G that are not incident to an edge in E, then G has a planar ω-decomposition D of width 2 with $V(D) = \{\{v\} : v \in V\} \cup \{\{v,w\} : vw \in E\}$ with no duplicate bags.*

Since the bags of D correspond to vertices and edges of G (with no duplicates) each vertex of G has spread $s(v) \leq \deg(v)$. Thus Lemmas 3.4 and 3.2 imply that every graph G with no K_5-minor has crossing number $\mathsf{cr}(G) \leq \sum_{v\in V(G)} \deg(v)^3$. This result represents a qualitative improvement over the $\mathcal{O}(\Delta(G)^2|V(G)|)$ bound in [54]. But we can do better. In particular, Lemmas 3.4 and 3.3 with $c=1$ imply that $\mathsf{cr}(G) \leq 4\sum_{vw\in E(G)} \deg(v)\deg(w)$. Thus Lemma A.1 implies:

Theorem 3.5. *Every graph G with no K_5-minor has crossing number $\mathsf{cr}(G) \leq 8\sum_{v\in V(G)} \deg(v)^2$.*

4 Interval Graphs and Chordal Graphs

A graph is *chordal* if every induced cycle is a triangle. An *interval graph* is the intersection graph of a set of intervals in $\mathbb{R}$. Every interval graph is chordal.

Theorem 4.1. *Every interval graph G has convex crossing number*

$$\mathsf{cr}^\star(G) \leq \tfrac{1}{2}(\omega(G)-2)\sum_{v\in V(G)} \deg(v)(\deg(v)-1) \leq (\omega(G)-2)(\omega(G)-1)(\Delta(G)-1)|V(G)|.$$

Proof. Jamison and Laskar [25] proved that G is an interval graph if and only if there is a linear order $\preceq$ of $V(G)$ such that if $u \prec v \prec w$ and $uw \in E(G)$ then $uv \in E(G)$. Orient the edges of G left to right in $\preceq$. Position $V(G)$ on a circle in the order of $\preceq$, with the edges drawn straight. Say edges xy and vw cross. Without loss of generality, $x \prec v \prec y \prec w$. Thus $vy \in E(G)$. Charge the crossing to vy. Say the out-neighbours of v are $w_1, \dots, w_d$. The in-neighbourhood of each w_i is a clique including v. Hence each w_i has at most $\omega(G) - 2$ in-neighbours to the left of v. Now v has $d - i$ neighbours to the right of w_i. Thus the number of crossings charged to vw_i is at most $(\omega(G) - 2)(d - i)$. Hence the number of crossings charged to outgoing edges at v is at most $\frac{1}{2}(\omega(G)-2)(d-1)d$. Therefore the total number of crossings is at most $\frac{1}{2}\sum_v(\omega(G)-2)(d_v-1)d_v$, where d_v is the out-degree of v. The other claims follow since $|E(G)| < (\omega(G) - 1)|V(G)|$. □

The *pathwidth* of a graph G is the minimum k such that G is a spanning subgraph of an interval graph G' with $\omega(G') \le k + 1$.

Theorem 4.2. *Every graph G with pathwidth k has convex crossing number* $\mathsf{cr}^\star(G) \le k^2 \cdot \Delta(G) \cdot |V(G)|$.

Proof. G is a spanning subgraph of an interval graph G' with $\omega(G') \le k + 1$. Apply the drawing algorithm in the proof of Theorem 4.1 to G'. Say edges xy and vw of G cross. Without loss of generality, $x \prec v \prec y \prec w$. Thus $vy \in E(G')$. Charge the crossing to vy. Now v has at most $\Delta(G)$ neighbours in G to the right of y. The in-neighbourhood of y is a clique in G' including v. Hence y has at most k neighbours to the left of v. Thus the number of crossings charged to vy is at most $k \cdot \Delta(G)$. Since G' has less than $k \cdot |V(G)|$ edges, the total number of crossings is at most $k^2 \cdot \Delta(G) \cdot |V(G)|$. □

Lemma 4.3. *Let D be an outerplanar decomposition of a graph G. Then G has a convex drawing such that if two edges e and f cross then some bag of D contains both an endpoint of e and an endpoint of f.*

Proof. Assign each vertex v of G to a bag $B(v)$ that contains v. Fix a crossing-free convex drawing of D. Replace each bag B of D by the set of vertices of G assigned to B. Draw the edges of G straight. Consider two edges vw and xy of G. Thus there is a path P in D between $B(v)$ and $B(w)$ and every bag in P contains v or w. Similarly, there is a path Q in D between $B(x)$ and $B(y)$ and every bag in Q contains x or y. Now suppose that vw and xy cross. Without loss of generality, the endpoints are in the cyclic order (v, x, w, y). Thus in the crossing-free convex drawing of D, the vertices $(B(v), B(x), B(w), B(y))$ appear in this cyclic order. Since D is crossing-free, P and Q have a bag X of D in common. Thus X contains v or w, and x or y. □

Theorem 4.4. *Every chordal graph G has convex crossing number* $\mathsf{cr}^\star(G) \le \sum_{vw \in E(G)} \deg(v)\deg(w)$.

Proof. It is well known that every chordal graph has a strong tree decomposition in which each bag is a clique. By Lemma 4.3, G has a convex drawing such that if two edges vw and xy of G cross then some bag B of D contains v or w, and x or y. Say B contains v and x. Since B is a clique, vx is an edge. Charge the crossing to vx. In every crossing charged to vx, one edge is incident to v and the other edge is incident to x. Since edges are drawn straight, no two edges cross twice. Thus the number of crossings charged to vx is at most $\deg(v)\deg(x)$. Hence the total number of crossings is as claimed. □

A k-tree is a chordal graph with maximum clique size $k + 1$. Every subgraph on n vertices of a k-tree has less than kn edges, thus by Lemma A.2 and Theorem 4.4.

Theorem 4.5. *Every k-tree G has convex crossing number* $\mathsf{cr}^\star(G) \le 16k^2 \cdot \Delta(G) \cdot |V(G)|$.

5 Graphs Excluding a Fixed Minor

In this section we prove our main result (Theorem 1.3): for every graph H there is a constant $c = c(H)$, such that every H-minor-free graph G has a crossing number at most $c\Delta(G) \cdot |V(G)|$. The proof is based on Robertson and Seymour's rough characterization of H-minor-free graphs, which we now introduce. For integers $h \geq 1$ and $\gamma \geq 0$, Robertson and Seymour [46] defined a graph G to be *h-almost embeddable* in $\mathbb{S}_\gamma$ if G has a set X of at most h vertices (called *apices*) such that $G \setminus X$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_h$ such that:

- G_0 has an embedding in $\mathbb{S}_\gamma$.
- The graphs $G_1, \dots, G_h$ (called *vortices*) are pairwise disjoint.
- There are faces[7] $F_1, \dots, F_h$ of the embedding of G_0 in $\mathbb{S}_\gamma$, such that each $F_i = V(G_0) \cap V(G_i)$.
- If $F_i = (u_{i,1}, u_{i,2}, \dots, u_{i,|F_i|})$ in clockwise order about the face, then G_i has a strong $|F_i|$-path decomposition Q_i of width h, such that each vertex $u_{i,j}$ is in the j-th bag of Q_i.

Theorem 5.1. *For all integers $h \geq 1$ and $\gamma \geq 0$ there is a constant $k = k(h, \gamma) \geq h$, such that every graph G that is h-almost embeddable in $\mathbb{S}_\gamma$ has crossing number at most $k\Delta(G) \cdot |V(G)|$.*

Proof. Let X and $\{G_0, G_1, \dots, G_h\}$ be the parts of G as specified in the definition of h-almost embeddable graphs. Let $\Delta := \Delta(G)$ and $n := |V(G)|$. Start with an embedding of G_0 on $\mathbb{S}_\gamma$. For each $i \in \{1, \dots, h\}$, draw vortex G_i inside of the face F_i on $\mathbb{S}_\gamma$, as prescribed in Theorem 4.2. The resulting drawing of $G \setminus X$ in $\mathbb{S}_\gamma$ has at most $h^2\Delta n$ crossings. Replace each crossing by a dummy degree-4 vertex. The resulting graph G' has genus at most γ. By Theorem 1.1, $\mathsf{cr}(G') \leq c\sum_{v \in V(G')} \deg(v)^2 \leq c\sum_{v \in G \setminus X} \deg(v)^2 + c4^2h^2\Delta n$. Since $\mathsf{cr}(G \setminus X) \leq h^2\Delta n + \mathsf{cr}(G')$, $\mathsf{cr}(G \setminus X) \leq c\sum_{v \in G \setminus X} \deg(v)^2 + (c4^2 + 1)h^2\Delta n$.

Consider a drawing of $G \setminus X$ in the plane that achieves at most this many crossings. Add each vertex of X to the drawing at some arbitrary position and draw its incident edges to obtain a drawing of G. Since $|X| \leq h$, there are at most $h\Delta$ edges in G that are not in $G \setminus X$. Each such edge crosses at most $|E(G)|$ edges in the drawing of G. Thus $\mathsf{cr}(G) \leq \mathsf{cr}(G \setminus X]) + h\Delta|E(G)| \leq k\Delta(G)|V(G)|$. ☐

Let G_1 and G_2 be disjoint graphs. Suppose that C_1 and C_2 are cliques of G_1 and G_2 respectively, each of size k, for some integer $k \geq 0$. Let $C_1 = \{v_1, v_2, \dots, v_k\}$ and $C_2 = \{w_1, w_2, \dots, w_k\}$. Let G be a graph obtained from $G_1 \cup G_2$ by identifying v_i and w_i for each $i \in [1, k]$, and possibly deleting some of the edges v_iv_j. Then G is a *k-clique-sum* of G_1 and G_2 *joined* at $C_1 = C_2$. An ℓ-clique-sum for some $\ell \leq k$ is called a $(\leq k)$-clique-sum.

The following rough characterization of H-minor-free graphs is a deep theorem by Robertson and Seymour [46]; see the recent survey by Kawarabayashi and Mohar [26].

Theorem 5.2. (Graph Minor Decomposition Theorem [46]) *For every graph H there is a positive integer $h = h(H)$, such that every H-minor-free graph G can be obtained by $(\leq h)$-clique-sums of graphs that are h-almost embeddable in some surface in which H cannot be embedded.*

By the graph minor structure theorem, Theorem 1.3 is directly implied by the following theorem.

Theorem 5.3. *For all integers $h \geq 1$ and $\gamma \geq 0$ there is a constant $c = c(h, \gamma) \geq h$, such that every graph G that can be obtained by $(\leq h)$-clique-sums of graphs that are h-almost embeddable in $\mathbb{S}_\gamma$ has crossing number at most $c\Delta(G) \cdot |V(G)|$.*

[7] We equate a face with the set of vertices on its boundary.

The remainder of this section is dedicated to proving Theorem 5.3 for G. Let $\Delta := \Delta(G)$. Let U be the set of integers $\{1, 2, \ldots, |U|\}$, such that $\{G_i : i \in U\}$ is the set (of the minimum cardinality) of graphs such that for all $i \in U$, G_i is h-almost embeddable in $\mathbb{S}_\gamma$, and G is obtained by $(\leq h)$-clique-sums of graphs in the set. These graphs can be ordered $G_1, \ldots, G_{|U|}$, such that for all $j \geq 2$, there is a minimum integer $i < j$, such that G_i and G_j are joined at some clique C in the construction of G. We say G_j is a *child* of G_i, G_i is a *parent* of G_j, and $P_j := V(C)$ is the *parent clique* of G_j. We consider the parent clique of G_1 to be the empty set; that is, $P_1 = \emptyset$. This defines a rooted tree T with vertex set U where ij is an edge of T if and only if G_j is a child of G_i. Let U_i denote the set of children of i in T. Let T_i denote the subtree of T rooted in i. For $S \subset V(T)$, let $G[S]$ be the graph induced in G by $\bigcup\{V(G_\ell) : \ell \in S\}$. For example, for $S = \{i\}$, $G[S] = G_i$.

The proof outline is as follows. For each G_i, $i \in U$, we define an auxiliary graph K_i (closely related to G_i), such that $|E(K_i)| \leq \mathcal{O}(\sum_{v \in V(G_i)\setminus P_i} \deg_G(v))$. We draw each K_i in the plane with at most $f(h)\Delta|E(K_i)|$ crossings, where f is some function on h. We then join the drawings of $K_1, \ldots, K_{|U|}$ into a drawing of G, where the price of the joining is an additional $f(h)\Delta$ crossings on each edge of K_i, $i \in U$. Thus the crossing number of G is at most $f(h)\Delta \sum_{i\in U} |E(K_i)|$, which, by the above claim on the number of edges of K_i, is at most $f(h)\Delta \sum_{i\in U} \sum_{v\in V(G_i)\setminus P_i} \deg_G(v) \leq f(h)\Delta \sum_{v\in V(G)} \deg_G(v) \leq f(h)\Delta|E(G)| \leq f(h)\Delta|V(G)|$, which is the desired result.

Defining K_i. For each $i \in U$, let $G_i^- := G_i[V(G_i) \setminus P_i]$. Note that, for each $v \in V(G)$ there is precisely one value $t \in U$ for which $v \in G_t^-$. Thus $\{V(G_1^-), \ldots, V(G_{|U|}^-)\}$ is a partition of $V(G)$. For each $i \in U$, define K_i as follows. Start with G_i^-. For each child G_j of G_i (that is, for each $j \in U_i$), add a new vertex c_j to G_i^-. For each edge $vw \in G$ such that $v \in G_i^- \cap P_j$ (that is, $v \in P_j \setminus P_i$) and $w \in G_\ell^-$ where $\ell \in V(T_j)$, connect v and c_j by an edge. Subdivide that edge once and label the subdivision vertex by the triple $(v, w, ¶_{vw})$, where $¶_{vw}$ is a path in T from i to ℓ (thus, $¶_{vw} = (i, j, \ldots, \ell)$). The resulting graph is K_i. Note that for each v in G_i^-, $\deg_{K_i}(v) = \deg_{G\setminus P_i}(v)$.

Suppose that for each $i \in U$, we remove each c_j, $j \in U_i$, from K_i. Consider the union of the resulting graphs, over all $i \in U$. Suppose that, for each vertex labelled $(v, w, ¶_{vw})$ in the union, we connect $(v, w, ¶_{vw})$ and w by an edge. The resulting graph is a subdivision of G. This is the strategy that we will follow when constructing a drawing of G. Namely, first draw each K_i. Then take the union of all the drawings. Then remove all c_j's. Finally, to obtain a drawing of G, route each missing edge of G. In particular, for a missing edge between $(v, w, ¶_{vw})$ and w with $¶_{vw} = (i, j, \ldots, \ell)$, we route that edge from $(v, w, ¶_{vw})$ in the drawing of K_i, through the drawing of K_j, ..., to w in the drawing of K_ℓ.

We first prove that the number of edges in K_i is as claimed in the outline. In addition to the edges in $E(G_i^-)$, K_i contains two edges for each edge $vw \in E(G)$, such that $v \in G_i^-$ and $w \in G_\ell^-$, where $\ell \in V(T_i) \setminus i$. Thus $|E(K_i)| \leq 2\sum_{v\in G_i^-} \deg_G(v) = 2\sum_{v\in V(G_i)\setminus P_i} \deg_G(v)$.

Drawing K_i. For each G_i, let A_i denote the set of apex vertices of G_i that are not in P_i.

Lemma 5.4. *For each $i \in U$, the crossing number of K_i is at most $f(h)\Delta|E(K_i)|$.*

Proof. Remove all the vertices of A_i from K_i. We now prove that the resulting graph $K_i[V(K_i)\setminus A_i]$, or $K_i\setminus A_i$ for short, can be drawn in $\mathbb{S}_\gamma$, with at most $f(h)\Delta|E(K_i\setminus A_i)|$ crossings. That will complete the proof since Theorem 1.1 implies that $\mathsf{cr}(K_i \setminus A_i) \leq f(h)\Delta|E(K_i \setminus A_i)|$, the same way it did in the proof of Theorem 5.1. Then we add back each vertex of A_i to the drawing of $K_i \setminus A_i$ at some arbitrary position in the plane and draw its incident edges to obtain a drawing of K_i. As in the proof of Theorem 5.1, $\mathsf{cr}(K_i) \leq \mathsf{cr}(K_i \setminus A_i) + h\Delta|E(K_i)| \leq f(h)\Delta|E(K_i)|$.

Thus it remains to prove that $K_i \setminus A_i$ can be drawn in $\mathbb{S}_\gamma$, with at most $f(h)\Delta|E(K_i \setminus A_i)|$ crossings. $Q := G_i^- \setminus A_i$ is an apex-free h-almost embeddable graph on $\mathbb{S}_\gamma$, with parts $\{Q_0, Q_1, \ldots, Q_h\}$, where Q_0 is the subgraph of Q embedded in $\mathbb{S}_\gamma$ and $\{Q_1, \ldots, Q_h\}$ are its vortices. For each $j \in U_i$, let C_j denote the subgraph of $K_i \setminus A_i$ induced by c_j and the vertices at distance at most two from c_j. The vertices at distance 2 from c_j form a clique C which is the join clique of C_j and Q. It is simple to verify that C_j has a strong tree decomposition J of width $h+2$, where J is a rooted star whose root bag contains $C \cup \{c_j\}$, and for each $(v, w, \P_{vw}) \in C_j$, J contains a leaf bag with $\{w, c_j, (v, w, \P_{vw})\}$ if $w \in C$, otherwise w is in A_i and the leaf bag contains $\{c_j, (v, w, \P_{vw})\}$.

We now add the vortices and C_j's to Q_0 to obtain a drawing of $K_i \setminus A_i$ in $\mathbb{S}_\gamma$ while creating at most $f(h)\Delta|E(K_i \setminus A_i)|$ crossings in $\mathbb{S}_\gamma$.

For each $j \in U_i$, C_j is joined to a clique C of Q. If C contains a vertex v of Q_ℓ, where $\ell \in \{2, \ldots, h\}$, then each vertex of C is in Q_ℓ. In that case, we say that C_j belongs to F_ℓ. Otherwise, all the vertices of C are in Q_0. In that case, an extended version of the graph minor decomposition theorem (see, [14, 26]) states that, $|C| \leq 3$ and moreover, if $|C| = 3$, then the 3-cycle induced by C is a face in Q_0. In that case, we say that C_j belongs to face C (if $|C| \leq 2$ we assign C_j to any face of Q_0 incident to all the vertices of C).

Now consider a face F of Q_0, its vortex Q_F, and all C_j, $j \in U'_i \subseteq U_i$, that belong to F. Let F' be the graph induced in $K_i \setminus A_i$ by the union of all these vertices that belong to F. Consider a strong path decomposition P_F of $F \cup Q_F$, as defined by the h-almost embedding. If F has no vortex, then its strong path decomposition P_F is just a bag containing $|F| \leq 3$ vertices of F in it. For each $j \in U'_i$, the join clique C of C_j is in some bag of P_F. Join the decomposition P_F and J by adding an edge between that bag of P_F and the root of J. It is simple to verify that the resulting strong tree decomposition of F' can be converted into a strong path decomposition of width $h+3$. Thus by Theorem 4.2, F' can be drawn inside of F with at most $(h+3)^2\Delta|E(F')|$ crossings. Accounting for all the faces of Q_0 gives $f(h)\Delta|E(K_i \setminus A_i)|$ bound on the number of crossings in the resulting drawing of $K_i \setminus A_i$ in $\mathbb{S}_\gamma$, as required. □

In addition to having as few crossings as proved in Lemma 5.4, we will need a drawing of K_i that has the following extra properties.

Lemma 5.5. *For each $i \in U$, there is a drawing of K_i with at most $f(h)\Delta|E(K_i)|$ crossings such that:*

(1) No pair of vertices in K_i has the same x-coordinate;

(2) For each $j \in U_i$, there is a square[8] D_j such that $D_j \cap K_i = c_j$, and c_j is an internal point of the top side of D_j, and no vertex in $V(K_i) \setminus \{c_j\}$ has the same x-coordinate as any point of D_j; and

(3) For any two $j, t \in U_i$, there is no line parallel to the y-axis that intersects both D_j and D_t.

(4) Moreover, given a circular ordering σ_j of the edges incident to each vertex c_j in K_i, $j \in U_i$, there is a drawing of K_i that satisfies (1)–(3) *such that the circular ordering of the edges incident to each c_j respects σ_j.*

Proof. Apply Lemma 5.4 to K_i to obtain a drawing of K_i with at most $s := f(h)\Delta|E(K_i)|$ crossings. By an appropriate rotation, we may assume that condition (1) is satisfied immediately. Clearly, the edges incident to c_j can be bent without changing the number of crossings such that there is a small enough square D_j that satisfies all the properties imposed on D_j, as stated in (2). Similarly, condition (3) is satisfied by shrinking the squares further, if necessary.

Consider a disk C_j centered at c_j, such that the only vertex of K_i that intersect the disk is c_j and the only edges of K_i that intersect C_j are the edges incident to c_j. Order the edges around c_j with respect to σ_j by moving (that is, bending) the edges incident to c_j within $C_j \setminus D_j$. This may

[8]By a *square*, we mean a 4-sided regular polygon together with its interior.

introduce new crossings. Each new crossing point is in $C_j \setminus D_j$ and thus it occurs between a pair of edges incident to c_j. There are at most $h\Delta$ edges incident to c_j. Thus each edge incident to c_j gets at most $h\Delta$ new crossings. Therefore, the resulting drawing of K_i satisfies conditions (1)–(4) and has at most $s + f(h)\Delta|E(K_i)| \leq f(h)\Delta|E(K_i)|$ crossings. □

Joining the K_i's into a Drawing of G. We obtain a drawing of G from the union of the drawings of K_i, $i \in U$, as follows. Join the drawings of these graphs in the order determined by a breath-first search on T, as follows. For each G_i, consider a drawing of K_i together with the squares incident to its children, as defined in Lemma 5.5. For each $j \in U_i$, place the drawing of K_j strictly inside of the square D_j of K_i (while scaling the drawing of K_j, if necessary). Denote by K the resulting drawing of $\bigcup_i K_i$. This procedure introduces no new crossings, thus by Lemma 5.5, the number of crossings in K is at most $\sum_{i\in U} f(h)\Delta|E(K_i)|$.

We are now ready to define the ordering σ_j of edges around each vertex c_j, $j \in U \setminus \{1\}$. Consider an edge e_1 incident to c_j and $(v, w, \P_{vw})$, and an edge e_2 incident to c_j and $(a, b, \P_{ab})$. Define $e_1 \leq_{\sigma_j} e_2$ if the x-coordinate of w in K is less than the x-coordinate of b in K. If $w = b$, order e_1 and e_2 arbitrarily. Since no pair of vertices in K have the same x-coordinate, σ_j is a total order of the edges incident to c_j.

For each $j \in U \setminus \{1\}$, we may assume that the graph induced in K by c_j and its neighbours (the subdivision vertices), is a crossing-free star in K; that is, each edge of the star is not crossed by any other edge of K.

For each $i \in U$, remove each c_j, $j \in U_i$, from K. The subdivision vertices of K become degree-1 vertices. For each such subdivision vertex $(v, w, \P_{vw})$, where $\P_{vw} = (i, j, \dots, \ell)$ draw an edge from $(v, w, \P_{vw})$ to the point on the top side of the square D_j that has the same x-coordinate as w in K. Since $w \in G[T_j] \setminus P_j$, by construction $w \in D_j$, and thus such a point must exist on the top side of D_j. If w is an endpoint of $s \geq 2$ such edges, draw s points very close together on the top side of D_j and connect each of the s edges to one of the s points in the order σ_j. (In fact, imagine that these points are almost overlapping; that is, their x-coordinates are almost the same as that of w in K). Since the star incident to c_j is crossing-free in K, this can be done so that the resulting graph K_i^- has the same number of crossings as K_i. Label each point on the top side of D_j by the same label as the subdivision vertex it is adjacent to. (In fact, consider that point on the top side of D_j to be the subdivision vertex instead of the old one). Draw a line-segment between each subdivision vertex $(v, w, \P_{vw})$ on the top side of D_j and w. Call these segments *vertical segments*. This defines a drawing of G. We now prove that the number of crossings in G does not increase much compared to the number of crossings in K. Specifically, it increases by at most $f(h)\Delta \sum_{i\in U} |E(K_i)|$.

Note that Lemma 5.5 does not define the square D_1. Let D_1 be the whole plane. For each $i \in U$, let D_i^- be the region of the plane $D_i \setminus \{\bigcup_{j\in U_i} D_j\}$. Denote by d_i the number of crossings in the drawing of G restricted to D_i^-. Then $\mathsf{cr}(G) \leq \sum_i d_i$.

We now prove that for each $i \in U$, $d_i \leq f(h)\Delta|E(K_i)|$, which will complete the proof. Quantity d_i is at most the number of crossings in K_i^- plus the number of crossings caused by the vertical segments intersecting D_i^-. By construction (in particular, properties (2) and (3) of Lemma 5.5), each vertical segment that intersects D_i^- is a part of an edge that has one endpoint in G_s^- where $i \in V(T_s) \setminus s$ (that is, G_s^- is an ancestor of G_i^-) and its other endpoint is either in G_i^- (and, thus is K_i^-) or is in a descendent G_ℓ^- of G_i^-. Thus the number of vertical segments that cross D_i^- is at most $f(h)\Delta$. No pair of vertical segments cross in D_i^- due to their ordering. Thus each new crossing in D_i^- (that is, a crossing not present in the drawing of K_i^-) occurs between a vertical segment and an edge of K_i^-. Thus each edge of K_i^- accounts for at most $f(h)\Delta$ new crossings, and thus $d_i \leq f(h)\Delta|E(K_i^-)| \leq f(h)\Delta|E(K_i)|$, as desired. This completes the proof of Theorem 5.3.

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A Linear Bounding Functions

In this appendix we give some sufficient conditions for a graph to satisfy certain linear bounds on the crossing number.

Lemma A.1. *Let X be a class of graphs closed under taking subdivisions. Suppose that*

$$\mathsf{cr}(G) \leq c \sum_{vw \in E(G)} \deg(v)\deg(w)$$

for every graph $G \in X$. Then

$$\mathsf{cr}(G) \leq 2c \sum_{v \in V(G)} \deg(v)^2$$

for every graph $G \in X$.

Proof. Let $G \in X$. Let G' be the graph obtained from G by subdividing every edge once. By assumption, $G' \in X$ and

$$\begin{aligned}\mathsf{cr}(G') &\leq c \sum_{vw \in E(G')} \deg(v)\deg(w) \\ &= c \sum_{vw \in E(G)} (2\deg(v) + 2\deg(w)) \\ &= 2c \sum_{vw \in E(G)} (\deg(v) + \deg(w)) \\ &= 2c \sum_{v \in V(G)} \deg(v)^2.\end{aligned}$$

The result follows since $\mathsf{cr}(G) = \mathsf{cr}(G')$. □

We can also conclude a $\mathcal{O}(\Delta(G) \cdot |V(G)|)$ bound from $\sum_{vw \in E(G)} \deg(v)\deg(w)$.

Lemma A.2. *Let G be a graph with bounded arboricity. In particular, every subgraph of G on n vertices has at most kn edges. Then*

$$\sum_{vw \in E(G)} \deg(v)\deg(w) \;\leq\; 16k \cdot \Delta(G) \cdot |E(G)| \;\leq\; 16k^2 \cdot \Delta(G) \cdot |V(G)|.$$

Proof. Let $i, j \geq 0$ be integers. Let

$$\begin{aligned}\Delta_i &:= \Delta(G)/2^i \\ V_i &:= \{v \in V(G) : \Delta_{i+1} < \deg(v) \leq \Delta_i\} \\ n_i &:= |V_i| \\ E_{i,j} &:= \{vw \in E(G) : v \in V_i, w \in V_j\} \\ e_{i,j} &:= |E_{i,j}| \;.\end{aligned}$$

Let $S_i := \{j \geq 0 : n_j \leq n_i\}$. Thus

$$\sum_{vw \in E(G)} \deg(v)\deg(w) \;\leq\; \sum_{i \geq 0} \sum_{j \in S_i} \sum_{vw \in E_{i,j}} \deg(v)\deg(w)$$

$$
\begin{aligned}
&\le \sum_{i\ge 0}\sum_{j\in S_i} e_{i,j}\Delta_i\Delta_j \\
&\le k\sum_{i\ge 0}\sum_{j\in S_i}(n_i+n_j)\Delta_i\Delta_j \\
&\le 2k\sum_{i\ge 0}\sum_{j\ge 0} n_i\Delta_i\Delta_j \\
&\le 2k\sum_{i\ge 0} n_i\Delta_i\sum_{j\ge 0}\Delta_j \ .
\end{aligned}
$$

Since $\sum_{j\ge 0}\Delta_j \ < \ 2\cdot\Delta(G)$,

$$\sum_{vw\in E(G)} \deg(v)\deg(w) \ < \ 4k\cdot\Delta(G)\sum_{i\ge 0} n_i\Delta_i \ .$$

Observe that

$$2|E(G)| \ = \ \sum_{i\ge 0}\sum_{v\in V_i}\deg(v) \ > \ \sum_{i\ge 0} n_i\Delta_{i+1} \ = \ \tfrac{1}{2}\sum_{i\ge 0} n_i\Delta_i \ .$$

Thus

$$\sum_{vw\in E(G)} \deg(v)\deg(w) \ < \ 16k\cdot\Delta(G)\cdot|E(G)| \ .$$

□