The Number of Lines Tangent to Arbitrary Polytopes in $\mathbb{R}^3$

H. Brönnimann[†] O. Devillers[||] V. Dujmović[‡] H. Everett[§] M. Glisse[§]
X. Goaoc[§] S. Lazard[§] H.-S. Na[**] S. Whitesides[‡]

Abstract

We prove that the lines tangent to four possibly intersecting polytopes in $\mathbb{R}^3$ with n edges in total form $\Theta(n^2)$ connected components. In the generic case, each connected component is a single line, but our result still holds for arbitrary degenerate scenes. This improves the previously known $O(n^3 \log n)$ bound by Agarwal [1]. More generally, we show that a set of k polytopes with a total of n edges admits, in the worse case, $\Theta(n^2k^2)$ connected components of lines tangent to any four of these polytopes.

1 Introduction

Computing visibility relations in a 3D environment is a central problem of computer graphics (e.g. determining the view from a given point, identifying the set of blockers between two polygons, computing the umbra and penumbra cast by an area light source) and other engineering tasks (e.g. radio propagation simulation). Visibility is well-known to account for a significant portion of the cost of the overall computation in these applications, and consequently a large body of research is devoted to speeding up visibility computations through the use of data structures (see [7] for a survey, including space partitions, aspect graphs, and visual hulls).

One such structure, the visibility complex [10], which has been proposed for rendering [9], encodes visibility relations by partitioning the set of maximal free line segments. Unfortunately the complex is potentially enormous which has so far prevented its application in practice. Its size is intimately related to the number of lines tangent to four objects in the scene; for n triangles in $\mathbf{R}^3$, the complex can have size $\Theta(n^4)$ in the worst case [6], even when the triangles form a terrain (see Figure 1).

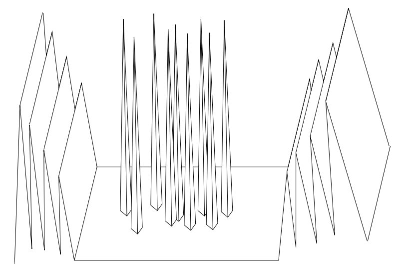

Figure 1: A terrain of size $O(n)$ with $\Omega(n^4)$ maximal non-occluded line segments tangent in four points.

There is evidence, both theoretical [5] and practical [8], that this bound is largely pessimistic in practical situations. The lower bound examples are carefully manufactured to exhibit the worst-case behavior, and often require an unrealistically large ratio between longest and shortest edges. On the other hand, for random scenes, Devillers et al. [5] prove that the expected size of the visibility complex is much smaller; for uniformly distributed balls the expected size is linear and for polygons or polyhedra of bounded aspect ratio it is at most quadratic. While these results are encouraging, most scenes are not random. In fact, most scenes have a lot of structure which we can exploit; a scene is typically represented by many triangles which form a much smaller number of convex patches. In [3], we proved, under a strong general position assumption, that if n triangles form k *disjoint* convex polyhedra, the number of tangents to four of the polyhedra is at most $O(n^2k^2)$ improving the previously known $O(n^3 \log n)$ bound of Agarwal [1].

[†]Polytechnic University, hbr@poly.edu
[‡]McGill University, {vida,sue}@mcgill.ca
[§]LORIA - INRIA Lorraine, {name}@loria.fr.
[||]INRIA Sophia-Antipolis, Olivier.Devillers@inria.fr.
[**]Soongsil University, hsnaa@computing.ssu.ac.kr

Our results. In this paper, we generalize the result of [3] in two ways. First, we consider polytopes that may intersect. We show that among k polytopes of total complexity n, the number of lines tangent to any four of them is either infinite or $\Theta(n^2k^2)$. The most surprising aspect of this result is that the bound (which is tight) is the same whether the polytopes intersect or not [3]. This is in sharp contrast to the 2D case, where the number of tangents of two convex polygons is always 4 if disjoint, and could be linear in the size of the polygons if they intersect.

Secondly we consider polytopes in arbitrary position: we drop all general position assumptions. Four such objects may admit an infinite number of common tangents which can be partitioned in line space into connected components. Two lines are said to be in the same *connected component* if and only if their corresponding points in line space (e.g., Plücker space) are in the same connected component of the set of points corresponding to all the lines tangent to any 4 among the k objects; equivalently one line can be continuously moved into the other while remaining tangent to at least four objects[1].

Our main results are the following.

Theorem 1 *The lines tangent to* 4 *polytopes in* $\mathbb{R}^3$ *with* n *edges in total form* $\Theta(n^2)$ *connected components. Moreover, if one of the polytopes has* m *edges, these bounds improve to* $\Theta(mn)$.

Theorem 2 *Given* k *polytopes in* $\mathbb{R}^3$ *with* n *edges in total, the lines (possibly occluded) tangent to any four of the polytopes form, in the worst case,* $\Theta(n^2k^2)$ *connected components.*

Our results hold for any set of polytopes including those that degenerate to a polygon, segment or point. The polytopes may intersect in any way; they may overlap or coincide and their faces may be triangulated.

To emphasize the importance of considering intersecting polytopes, observe that computer graphics scenes usually consist of not necessarily convex objects which can, however, be decomposed into sets of intersecting convex polyhedra.

[1]Notice that the set of objects the line is tangent to might change during the motion.

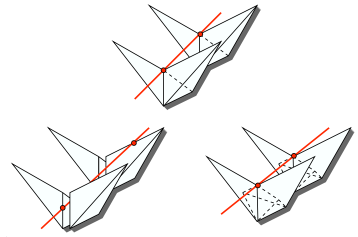

Figure 2: Decompositions into interior-disjoint vs. intersecting polytopes yield different tangents.

Notice that simply decomposing these objects into convex polyhedra with disjoint interiors may yield new tangents which were not present in the original scene; indeed a line tangent to two polytopes along a shared face is not always tangent to their union. Moreover, the decomposition of a polyhedron into interior-disjoint polytopes may have a much higher complexity than a decomposition into intersecting polytopes.

The importance of considering polytopes in arbitrary position comes from the fact that graphics scenes are full of degeneracies both in the sense that four polytopes may admit infinitely many tangents and that polytopes may share edges or faces. Note that we could not find a perturbation argument that guarantees the preservation of all connected components of tangents and we do not believe it is a simple matter.

The paper is organized as follows. We prove the upper bounds of Theorems 1 and 2 in Sections 2 and 3, and the lower bounds in Section 4.

2 Main lemma

We prove in this section the following lemma which yields the upper bounds in Theorem 1. This lemma is also fundamental for the proof of the upper bound of Theorem 2, which we provide in Section 3.

Consider three polytopes $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ in $\mathbb{R}^3$, with p, q, and r edges, respectively, and let e be an edge of a fourth polytope $\mathbf{S}$.

Main Lemma *The lines intersecting* e *and tangent to* $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ *and* $\mathbf{S}$ *form* $O(p+q+r)$ *connected components.*

The upper bounds of Theorem 1 follow from the Main Lemma. Indeed, if s denotes the number of edges of $\mathbf{S}$, by summing over all these edges, the number of connected components of lines tangent to the four polytopes is $O(s(p+q+r)) = O(n^2)$. Moreover, if one of the polytopes has m edges, choosing $\mathbf{S}$ to be this polytope yields that the lines tangent to the four polytopes form $O(mn)$ connected components.

The actual proof of the Main Lemma is rather complicated because it handles polytopes which may intersect in any way. In addition, the proof handles all the degenerate cases. To assist the reader, we first give an overview of the proof.

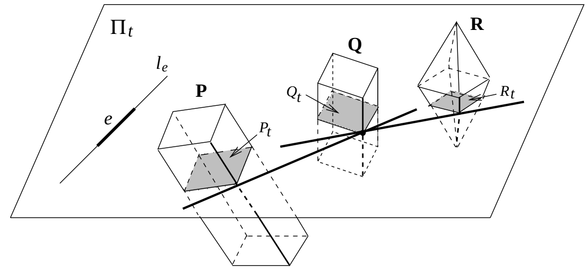

Figure 3: Plane Π_t contains edge e and intersects polytopes $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ in polygons P_t, Q_t, and R_t, respectively.

2.1 Proof overview

The proof is inspired by a method which was, to our knowledge, first used in [2] (see also Schiffenbauer's survey [11]).

We sweep the space with a plane Π_t rotating about the line containing e. The sweep plane intersects the three polytopes $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ in three, possibly degenerate or empty, convex polygons denoted P_t, Q_t, and R_t, respectively (see Figure 3).

During the sweep, we track the *bitangents*, that are the lines tangent to P_t and Q_t, or to Q_t and R_t. As the sweep plane rotates, the three polygons deform and the bitangents move accordingly. Every time two bitangents become aligned during the sweep, their supporting line is tangent to $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$. A bitangent to two polygons, say P_t and Q_t, goes through a set of vertices of these polygons. The edges of $\mathbf{P}$ and $\mathbf{Q}$ that contain these vertices are called *support edges* of the bitangent. (For the purpose of this overview we ignore the case where bitangents goes through vertices of $\mathbf{P}$ or $\mathbf{Q}$).

In any given instance of the sweep plane Π_t, we consider the pairs of bitangents (one involving P_t and Q_t, and the other Q_t and R_t) that share a vertex of Q_t (see Figure 3). The sets of edges that consist of e and the support edges of two such bitangents are called *candidate tuples.* A candidate tuple contains at least 4 edges, e and at least one edge from each polytope; most of the time, it contains exactly 4 edges.

A candidate tuple induced by an instance of the sweep plane changes as the plane rotates only when a support edge of a bitangent changes. We define *critical planes* in such a way that the support edges of the bitangents do not change as the sweep plane rotates in between two consecutive critical planes. As the sweep plane rotates, the support edges of a bitangent change if a support edge starts or ceases to be swept, or if, during its motion, the bitangent becomes tangent to one of the polygons along an edge (see Figure 4). We thus define two types of critical planes. An instance of the sweep plane is called *V-critical* if it contains a vertex of one of the polytopes, and *F-critical* if it contains a line that lies in the plane containing a face of one of the polytopes and is tangent to another of the polytopes (see Figure 4). We will show that the number of critical planes is $O(p+q+r)$.

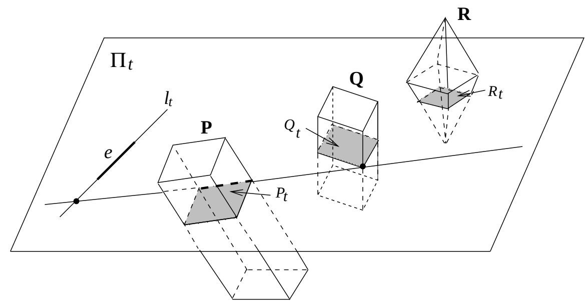

Figure 4: A bitangent to P_t and Q_t is tangent to P_t along an edge. The plane Π_t is F-critical.

Notice that when the polytopes intersect there may exist a linear number of bitangents in an instance of the sweep plane; two intersecting convex polygons may admit a linear number of bitangents. Thus there can be a linear number of candidate tuples induced by any instance of the sweep plane and the linear number of critical planes leads to a quadratic bound on the total number of distinct candidate tuples. In the detailed proof of the lemma (Section 2.2), we amortize the count of candidate tuples over all the critical planes to get a linear bound on the number of distinct candidate tuples and thus on the number of connected components of lines intersecting e and tangent to $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$.

2.2 Proof of the Main Lemma

We now give a detailed proof of the Main Lemma for the general case of possibly intersecting polytopes in any configuration.

Since any line tangent to a polytope will remain tangent whether or not the faces are triangulated (or otherwise subdivided), we assume without loss of generality that *no two faces of* $\mathbf{P}$ *are coplanar, and similarly for* $\mathbf{Q}$, $\mathbf{R}$ *and* $\mathbf{S}$.

We can also assume without loss of generality that $\mathbf{P}$*,* $\mathbf{Q}$*,* $\mathbf{R}$ *and* $\mathbf{S}$ *have non-empty interior.* Indeed, if, say, $\mathbf{P}$ has empty interior it can be "inflated" into a polytope $\mathbf{P}'$ so that all discrete tangents to $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ are also tangent to $\mathbf{P}'$, and all other connected component of tangents to $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ contains at least one line tangent to $\mathbf{P}'$. Such a modification only increases the number of connected components of common tangent lines and can be made by increasing the number of edges of each polytope by at most a multiplicative factor 2 (for inflating a polygon) and an additive factor 6 (for inflating a point or a segment).

Let l_e be the line containing e and let Π_t denote the sweep plane parameterized by $t \in [0, \pi]$ such that Π_t contains the line l_e for all t and $\Pi_0 = \Pi_\pi$. Each plane Π_t intersects the three polytopes $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ in three, possibly degenerate or empty, convex polygons, P_t, Q_t, and R_t, respectively (see Figure 3).

A line intersecting e and tangent to $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ is called a *generic tangent line* if it is tangent to P_t, Q_t, and R_t in some plane Π_t. Otherwise it is called a *nongeneric tangent line.* A nongeneric tangent line properly[2] intersects P_t, Q_t, or R_t in some plane Π_t. In this case P_t, Q_t, or R_t is a face or an edge of $\mathbf{P}$, $\mathbf{Q}$, or $\mathbf{R}$ lying in Π_t, a degenerate situation.

We bound in the following subsection the number of generic tangent lines. Due to lack of space the proof for non-generic tangent lines is deferred to Appendix C.

Generic tangent lines

We start with some definitions. We call *support vertex* of a line, a vertex of $\mathbf{P}$, $\mathbf{Q}$, or $\mathbf{R}$ that lies on the line. We call *support edge* of a line, an edge of $\mathbf{P}$, $\mathbf{Q}$, or $\mathbf{R}$ that intersects the line and has no endpoint on it. A *support* is a support edge or support vertex.

For any t, let a (P_t, Q_t)*-tuple* be the nonordered set of all supports in $\mathbf{P}$ and $\mathbf{Q}$ of one of the bitangents to polygons P_t and Q_t in Π_t. Note that a (P_t, Q_t)-tuple consists of exactly one edge of $\mathbf{P}$ and one edge of $\mathbf{Q}$, except when the corresponding bitangent is tangent to $\mathbf{P}$ or $\mathbf{Q}$ along a face or at a vertex; then the (P_t, Q_t)-tuple consists of one or two supports of $\mathbf{P}$ and one or two supports of $\mathbf{Q}$. We define similarly the (Q_t, R_t)*-tuples.*

We say that a (P_t, Q_t)-tuple is *maximal for some* t if it is not contained in any other (P_t, Q_t)-tuple, for the same t. Note that a (P_t, Q_t)-tuple u is non-maximal for some t only if the union of the edges and vertices of u intersects Π_t in one single point.

For any t, let a (P_t, Q_t, R_t)*-tuple* be the union of a (P_t, Q_t)-tuple and a (Q_t, R_t)-tuple that share at least an edge or vertex of $\mathbf{Q}$. A (P_t, Q_t, R_t)-tuple is maximal for some t if it is not contained in any other (P_t, Q_t, R_t)-tuple, for the same t. Note that a (P_t, Q_t, R_t)-tuple typically consists of three edges, one from each polytope, and consists, in all cases, of one or two supports of $\mathbf{P}$, of $\mathbf{R}$, and one, two or three supports of $\mathbf{Q}$.

Lemma 3 *The set of supports of a generic tangent line is a* (P_t, Q_t, R_t)*-tuple for some* t.

Proof: Any generic tangent line ℓ is tangent in Π_t to P_t, Q_t, and R_t for some value t. Thus the set of supports of ℓ in $\mathbf{P}$ and $\mathbf{Q}$ contains a (P_t, Q_t)-tuple and the set of supports of ℓ in $\mathbf{Q}$ and $\mathbf{R}$ contains a (Q_t, R_t)-tuple. Moreover these two tuples agree on the edges and vertices of $\mathbf{Q}$,

[2] A line *properly* intersects a polygon if and only if it intersects its relative interior.

thus their union is a (P_t, Q_t, R_t)-tuple. □

We now define the *critical planes* Π_t in such a way that the set of (P_t, Q_t, R_t)-tuples is invariant for t ranging strictly in between two consecutive critical values (see Lemma 5). We introduce two types of critical planes: the *V-critical* and *F-critical planes.*

A plane Π_t is *V-critical* if it contains a vertex of $\mathbf{P}$, $\mathbf{Q}$, or $\mathbf{R}$, not[3] on l_e. A plane Π_t is *F-critical* relative to polytopes $(\mathbf{P}, \mathbf{Q})$ if (see Figure 4) it contains a line ℓ such that

(i) ℓ lies[4] in a plane $\Psi \neq \Pi_t$ containing a face of $\mathbf{P}$, and

(ii) ℓ is tangent in Ψ to polygon[5] $\mathbf{Q} \cap \Psi$ at some point not on l_e.

F-critical planes relative to $(\mathbf{Q}, \mathbf{P})$, $(\mathbf{Q}, \mathbf{R})$, and $(\mathbf{R}, \mathbf{Q})$ are defined similarly. A plane Π_t is *F-critical* if it is F-critical relative to polytopes $(\mathbf{P}, \mathbf{Q})$, $(\mathbf{Q}, \mathbf{P})$, $(\mathbf{Q}, \mathbf{R})$, or $(\mathbf{R}, \mathbf{Q})$.

The values of t corresponding to critical planes Π_t are called *critical values.* We call *V-critical* and *F-critical events* the couples (t, o) where t is a critical value and o is a vertex or line depending on the type of critical event. In a V-critical event, o is a vertex of $\mathbf{P}$, $\mathbf{Q}$, or $\mathbf{R}$ that belongs to $\Pi_t \setminus l_e$. In an F-critical event, o is a line lying in some plane Π_t and satisfying Conditions (i-ii) above. A *critical event* is a V-critical or F-critical event.

Lemma 4 *There are at most $\frac{2}{3}(p+q+r)$ V-critical events and $\frac{4}{3}(p+2q+r)$ F-critical events.*

Proof: The number of V-critical events is at most the total number of vertices of $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$, and hence is less than two thirds the total number of edges of $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$.

We now count the number of F-critical events relative to polytopes $(\mathbf{P}, \mathbf{Q})$. Let Ψ be a plane containing a face of $\mathbf{P}$. Let ℓ be a line lying in some plane Π_t and satisfying with plane Ψ Conditions (i-ii). Plane Ψ does not contain l_e because otherwise both l_e and ℓ lie in Ψ and $\Pi_t \neq \Psi$, thus $\ell = l_e$ cannot satisfy Conditions (ii). Furthermore ℓ intersects l_e or is parallel to it since ℓ lies in Π_t. Thus if $\Psi \cap l_e$ is a point then ℓ intersects it, and otherwise $\Psi \cap l_e = \emptyset$ and ℓ is parallel to l_e.

There are at most two lines ℓ in plane Ψ going through $\Psi \cap l_e$ if this intersection is a point and parallel to l_e otherwise, and tangent to $\mathbf{Q} \cap \Psi$ at some point not on l_e. Moreover each such line is contained in a unique plane Π_t, for $t \in [0, \pi]$, since $\ell \neq l_e$ (ℓ contains a point not on l_e). Hence, a face of $\mathbf{P}$ generates at most two F-critical events relative to $(\mathbf{P}, \mathbf{Q})$. Therefore the number of critical events relative to $(\mathbf{P}, \mathbf{Q})$ is at most $\frac{4}{3}p$ since the number of faces of a polytope is at most two thirds the number of its edges. Hence the number of critical events relative to $(\mathbf{P}, \mathbf{Q})$, $(\mathbf{Q}, \mathbf{P})$, $(\mathbf{Q}, \mathbf{R})$ and $(\mathbf{R}, \mathbf{Q})$ is at most $\frac{4}{3}(p + 2q + r)$. □

We now prove that the critical planes have the desired property. Let u_e be the set of supports in $\mathbf{P}$ and $\mathbf{Q}$ of l_e.

Lemma 5 *If $u \neq u_e$ is a maximal*[6] *(P_t, Q_t)-tuple for some but not all t in every open neighborhood of t^*, then t^* is a critical value.*

Moreover, there exist a V-critical event (t^*, v) or a F-critical event (t^*, m) such that u contains v or an edge with endpoint v, or u is contained in the set of supports of m.

Proof: Let $\mathcal{N}$ denote an open neighborhood of t^*. We require $\mathcal{N}$ to be sufficiently small in various places in the proof; the reader should thus read $\mathcal{N}$ as "*any sufficiently small neighborhood of t^**". Let t_0 and t_1 denote the endpoint of $\mathcal{N}$ (t_0 and t_1 are not fixed) and $\mathcal{N}^\star$ denote $\mathcal{N} \setminus \{t^*\}$.

It is sufficient for proving the lemma to suppose that u is a maximal (P_t, Q_t)-tuple, for $t = t^*$ or all $t \in (t^*, t_1)$, but not for all $t \in \mathcal{N}$. Indeed, assume that Lemma 5 holds in these two

[3] This constraint ensures that the number of V-critical planes is finite even in degenerate configurations.

[4] For simplicity, we do not require that ℓ is tangent to $\mathbf{P}$; this leads to overestimating the number of common tangents to $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ but only by an asymptotically negligible amount.

[5] Note that not all lines in Ψ tangent to $\mathbf{Q}$ are tangent to the polygon $\mathbf{Q} \cap \Psi$ when that polygon is a face or edge of $\mathbf{Q}$ lying in Ψ.

[6] For simplicity, we prove Lemma 5 under the assumptions that u is maximal and distinct from u_e; note however that one could prove the result without these assumptions.

cases. Then the only case for which the lemma does not directly hold is if there exists, in every open neighborhood of t^*, infinitely many maximal intervals (open or closed and possibly of zero length) in which either u is a maximal (P_t, Q_t)-tuple for all t or u is not a maximal (P_t, Q_t)-tuple for all t. However, in this case, our assumption yields that each endpoint of these intervals is a critical value, contradicting Lemma 4.

We prove a series of claims that yields the result. Indeed, in the cases where the result does not trivially hold, we prove the existence of a line m in Π_{t^*} (Claim C) such that (i) m lies in a plane $\Psi \neq \Pi_{t^*}$ containing a face of $\mathbf{P}$ (Claim D), and (ii) m is tangent in Ψ to the polygon $\mathbf{Q} \cap \Psi$ at some point not on l_e (Claim E); moreover, the set of supports of m contains u (Claim C). This proves that Π_{t^*} contains a line m whose set of supports contains u and such that (t^*, m) is an F-critical event, which concludes the proof.

We can assume that *u contains no vertex v, or edge with endpoint v, such that v lies on $\Pi_{t^*} \setminus l_e$* because otherwise (t^*, v) is a V-critical event such that u contains v or an edge with endpoint v, which concludes the proof.

We only consider in the following supports in $\mathbf{P}$ and in $\mathbf{Q}$; polytope $\mathbf{R}$ plays no role.

We start by proving two preliminary facts. Because of lack of space, we consider here the general position assumption that no two edges of two distinct polytopes, among $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{S}$, are coplanar. This simplifies the proofs of Claims A and E; we give in Appendix A a complete proof of these two claims.

Claim A *Each edge and vertex of u intersects Π_t in exactly one point (possibly on l_e), for all $t \in \mathcal{N}$.*

Moreover, the union of the edges and vertices of u intersects Π_t in at least two distinct points for all $t \in \mathcal{N}$ if u is a maximal (P_{t^*}, Q_{t^*})-tuple, and for all $t \in \mathcal{N}^\star$ otherwise.

Proof of Claim A: We have assumed that u contains no vertex v, or edge with endpoint v, such that v lies on $\Pi_{t^*} \setminus l_e$. It follows that each edge and vertex of u intersects Π_t in at least one point for all $t \in \mathcal{N}$. Moreover, each edge of u either lies in l_e or intersects Π_t in exactly one point, for all $t \in \mathcal{N}$. However, no edge of u lies in l_e because otherwise, if an edge x of say $\mathbf{P}$ belongs to u, then any line tangent to P_t in Π_t and intersecting x contains an endpoint of x which is a vertex of $\mathbf{P}$; thus, by definition, u does not contain x but one of its endpoints. Hence each edge and vertex of u intersects Π_t in exactly one point for all $t \in \mathcal{N}$.

The second statement of the claim is trivial under the general position assumption we considered because of the lack of space. □

Claim B *If u is a maximal (P_{t^*}, Q_{t^*})-tuple then u consists of at least three edges or vertices.*

Proof of Claim B: Suppose for a contradiction that u consists of only two edges or vertices. It follows from Claim A that they intersect Π_t in exactly two distinct points for all $t \in \mathcal{N}$. Thus there exists for all $t \in \mathcal{N}$ a unique line m_t in Π_t whose set of supports contains u; moreover m_t is continuous in terms of t. Since u is a (P_{t^*}, Q_{t^*})-tuple, the set of supports of m_{t^*} is u. Thus, for all t in any sufficiently small $\mathcal{N}$, the set of supports of m_t is u. It follows, since m_{t^*} is tangent to P_{t^*} and Q_{t^*}, that m_t is tangent to P_t and Q_t for all $t \in \mathcal{N}$.

Hence, for all $t \in \mathcal{N}$, line m_t, whose set of supports is u, is tangent to P_t and Q_t in Π_t. Thus u is a (P_t, Q_t)-tuple for all $t \in \mathcal{N}$. Moreover, m_t is the unique line in Π_t whose set of supports contains u, thus u is a maximal (P_t, Q_t)-tuple for all $t \in \mathcal{N}$, contradicting the hypotheses of the lemma. □

Claim C *There exists a line m in Π_{t^*} whose set of supports contains u that is tangent to P_{t^*} and Q_{t^*} along an edge of one of them, say of P_{t^*}.*

Proof of Claim C: Consider first the case where u is a (P_{t^*}, Q_{t^*})-tuple. There exists in Π_{t^*} a line m tangent to P_{t^*} and Q_{t^*} whose set of supports is u. By Claim B, the set u of supports of m contains at least two edges or vertices of $\mathbf{P}$ (or of $\mathbf{Q}$). Furthermore, the supports of m in one polytope intersect Π_{t^*} in distinct points (by definition of supports). Thus m intersects

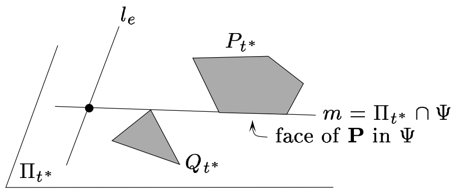

Figure 5: m is tangent to $\mathbf{P}$ along a face in plane $\Psi \neq \Pi_{t^*}$.

P_{t^*} (or Q_{t^*}) in at least two distinct points and is tangent to P_{t^*} and Q_{t^*}. The result follows since P_{t^*} (and Q_{t^*}) is convex.

Suppose now that u is a (P_t, Q_t)-tuple for all $t \in (t^*, t_1)$. Then, for all $t \in (t^*, t_1)$, there exists a line in Π_t tangent to P_t and Q_t and whose set of supports is u. Moreover, by Claim A, this line is unique for each $t \in (t^*, t_1)$ and varies continuously in terms of $t \in (t^*, t_1)$. When t tends to t^*, the line tends to a line m in Π_{t^*} which is tangent to P_{t^*} and Q_{t^*} and whose set of supports contains u. If its set of supports strictly contains u then m is tangent to P_{t^*} and Q_{t^*} along an edge of one of them because the polygons are convex. Otherwise, u is a (P_{t^*}, Q_{t^*})-tuple, a case which we already considered. $\square$

Claim D *Line m lies in a plane $\Psi \neq \Pi_{t^*}$ containing a face of* $\mathbf{P}$.

Proof of Claim D: By Claim C, m contains an edge of P_{t^*}; see Figure 5. This edge either intersects the relative interior of some face of $\mathbf{P}$ in which case we take Ψ to be the plane containing that face, or it is an edge of $\mathbf{P}$ in which case we take Ψ to be a plane, different from Π_{t^*}, containing one of the two faces of $\mathbf{P}$ incident to that edge. $\square$

Claim E *Line m is tangent to $\mathbf{Q} \cap \Psi$ at some point not on l_e.*

Proof of Claim E: By Claim C, m is tangent to Q_{t^*} and thus to $\mathbf{Q}$. It follows that m is tangent to $\mathbf{Q} \cap \Psi$ or properly intersects it, in which case $\mathbf{Q} \cap \Psi$ is a face or an edge of $\mathbf{Q}$. If m does not satisfy the claim, it is either tangent to $\mathbf{Q} \cap \Psi$ on l_e, or is tangent to $\mathbf{Q}$ along a face or an edge in Ψ and properly intersects it. In both cases, two edges, one in $\mathbf{Q}$ and one in $\mathbf{P}$ or e,

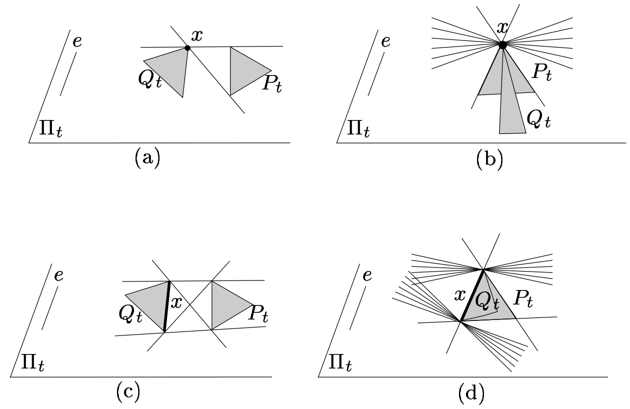

Figure 6: Lines tangent to P_t in Π_t and to the edge or vertex x of $\mathbf{Q}$.

are coplanar, contradicting the general position assumption we considered because of the lack of space. $\square$

This concludes the proof of Lemma 5. $\square$

Let x be any edge or vertex of $\mathbf{P}$ or $\mathbf{Q}$, and $\mathcal{I}$ be any interval of $\mathbb{R}/\pi\mathbb{Z}$ (open or closed, reduced to a point or not).

Lemma 6 *At most 2 distinct sets contain x and are maximal (P_t, Q_t)-tuples for all t in $\mathcal{I}$.*

Proof: Let $\tilde{t} \in \mathcal{I}$. If x does not intersect $\Pi_{\tilde{t}}$ then no set contains x and is a (P_t, Q_t)-tuple for all t in $\mathcal{I}$. If x intersects $\Pi_{\tilde{t}}$ in one point then there is, in general, at most two lines in $\Pi_{\tilde{t}}$ going through x and tangent to $P_{\tilde{t}}$ and $Q_{\tilde{t}}$ (see Figure 6(a)); in all cases there are at most 3 distinct sets of supports (in $\mathbf{P}$ and $\mathbf{Q}$) of bitangents to $P_{\tilde{t}}$ and $Q_{\tilde{t}}$ containing x (see Figure 6(b)); however only 2 of these $(P_{\tilde{t}}, Q_{\tilde{t}})$-tuples are maximal. If x intersects $\Pi_{\tilde{t}}$ in more than one points, it is an edge lying in $\Pi_{\tilde{t}}$. Then any line in $\Pi_{\tilde{t}}$ intersecting x and tangent to $P_{\tilde{t}}$ and $Q_{\tilde{t}}$ contains an endpoint of x and thus x belongs to no $(P_{\tilde{t}}, Q_{\tilde{t}})$-tuple.

Hence at most 2 distinct sets contain x and are maximal (P_t, Q_t)-tuples for $t = \tilde{t}$ and thus for all t in $\mathcal{I}$. $\square$

Lemma 7 *The number of distinct (P_t, Q_t, R_t)-tuples, for t ranging in $[0, \pi]$, is $O(p+q+r)$.*

Proof: In order to count the number of distinct (P_t, Q_t, R_t)-tuples, we charge each maximal (P_t, Q_t, R_t)-tuple to a critical event or a polytope edge or vertex. We then show that each critical event, edge or vertex is charged by at most a constant number of distinct maximal (P_t, Q_t, R_t)-tuples. It then follows from Lemma 4 that there are $O(p+q+r)$ distinct maximal (P_t, Q_t, R_t)-tuples. This implies the results since a (P_t, Q_t, R_t)-tuple consists of at most two supports of $\mathbf{P}$, at most three supports of $\mathbf{Q}$, and at most two supports of $\mathbf{R}$, and thus contains at most $3 \times 7 \times 3$ distinct subsets that can be a (P_t, Q_t, R_t)-tuple.

Let s be a maximal (P_t, Q_t, R_t)-tuple and let $\mathcal{I}$ be a maximal connected subset of $\mathbb{R}/\pi\mathbb{Z}$ such that s is a maximal (P_t, Q_t, R_t)-tuple for all $t \in \mathcal{I}$. Let u be the maximal (P_t, Q_t)-tuple and u' the maximal (Q_t, R_t)-tuple whose union is s and share at least an edge of vertex of $\mathbf{Q}$.

If $\mathcal{I} = \mathbb{R}/\pi\mathbb{Z}$ then u is maximal (P_t, Q_t)-tuple for all $t \in [0, \pi]$. Since u is maximal, the union of the edges and vertices of u intersects Π_t in at least two distinct points. Moreover, each edge and vertex of u intersects Π_t for all $t \in [0, \pi]$ and thus intersects l_e. It follows that l_e is the only line in Π_t whose set of supports is u, for any $t \in [0, \pi]$. Similarly for u'. Thus s is the set of supports of l_e. We assume in the following that $\mathcal{I} \neq \mathbb{R}/\pi\mathbb{Z}$, that s is not the set of supports of l_e, and add one to the maximum number of distinct sets of maximal (P_t, Q_t, R_t)-tuples.

Interval $\mathcal{I}$ is thus a non-empty interval of $\mathbb{R}/\pi\mathbb{Z}$; it can be open or closed, reduced to a point or not. Let w_0 and w_1 denote the endpoints of $\mathcal{I}$.

If s contains a vertex v, or an edge x with endpoint v, such that v lies in $\Pi_{w_i} \setminus l_e$, for $i = 0$ or 1, then we charge s to vertex v or edge x. Otherwise, we charge s to a F-critical event (w_i, m) where m is a line in Π_{w_i} whose set of supports contains u or u'; such an F-critical event exists by Lemma 5.

We prove in the two following claims that each edge and vertex of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$, and each F-critical event is charged by at most a constant number of distinct maximal (P_t, Q_t, R_t)-tuples. As mentioned before, that will imply the result.

Claim A *Any edge or vertex of* $\mathbf{P}$, $\mathbf{Q}$, *or* $\mathbf{R}$ *is charged by at most* $O(1)$ *distinct sets of maximal* (P_t, Q_t, R_t)*-tuples.*

Proof of Claim A: Let x be an edge or vertex of $\mathbf{P}$, $\mathbf{Q}$, or $\mathbf{R}$ charged by k distinct maximal (P_t, Q_t, R_t)-tuples $s_1, \ldots, s_k$. Let t_0 and t_1 be the at most two values such that Π_{t_i} contains an endpoint of x not on l_e. By the charging scheme, each s_i is a maximal (P_t, Q_t, R_t)-tuple for all t in an interval $\mathcal{I}$ that is either reduced to t_0 or t_1, or contains an open interval with one endpoint equal to t_0 or t_1. It follows that each s_i is a maximal (P_t, Q_t, R_t)-tuple for all t in at least one of four intervals, $\{t_0\}$, $\{t_1\}$, and two open intervals having t_0 and t_1 has endpoint; denote $\mathcal{I}_1, \ldots, \mathcal{I}_4$ these four intervals.

By Lemma 6, at most 2 sets contain x and are maximal (P_t, Q_t)-tuples for all t in $\mathcal{I}_i$. The same result holds for (Q_t, R_t)-tuples, thus at most 4 sets contain x and are maximal (P_t, Q_t, R_t)-tuples for all t in $\mathcal{I}_i$, for each $i = 1, \ldots, 4$. Hence any edge or vertex is charged by at most 16 distinct sets of maximal (P_t, Q_t, R_t)-tuples. □

Claim B *Any F-critical event is charged by at most* $O(1)$ *distinct sets of maximal* (P_t, Q_t, R_t)*-tuples.*

Proof of Claim B: Let s be a (P_t, Q_t, R_t)-tuple charging a F-critical event (t^*, m) and define as before u and u'. By the charging scheme, the set of supports of m contains u or u' (or both); suppose without loss of generality that it contains u. The set of supports of m contains at most two supports of $\mathbf{P}$ and at most two supports of $\mathbf{Q}$. Since u contains at least one edge or vertex of $\mathbf{P}$ and of $\mathbf{Q}$, there are at most 3^2 choices for u.

By the charging scheme, all the distinct sets of (P_t, Q_t, R_t)-tuples that are charged to (t^*, m) are maximal (P_t, Q_t, R_t)-tuples for all t in at least one of 3 intervals, $\{t^*\}$ and two open intervals having t^* has endpoint; denote $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ these intervals. It follows from Lemma 6 that, for each edge or vertex x of $\mathbf{Q}$ in u, at most 2 sets contain x and are maximal (Q_t, R_t)-tuples for all t in $\mathcal{I}_i$. There are at most 2 choices for x and 3 for i. Hence any F-critical event (t^*, m)

is charged by at most $2^2 \times 3^3$ distinct sets of maximal (P_t, Q_t, R_t)-tuples. $\square$

This concludes the proof of Lemma 7. $\square$

We now show in the next two lemmas that there are at most $O(p+q+r)$ connected components of generic tangent lines.

Lemma 8 *The set of transversals to e and to $O(1)$ edges or vertices of $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$, with at least one edge or vertex of each of the polytopes, contains at most $O(1)$ connected components of tangents to $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$.*

Proof: The set of transversals to $O(1)$ segments (possibly of zero length) consists of $O(1)$ connected components of complexity $O(1)$ [4]. The result directly follows if the set of transversals is finite. We omit here the proof in the case where this set is not zero-dimensional. We give a complete proof in Appendix B $\square$

Proposition 9 *The generic tangent lines form $O(p+q+r)$ connected components.*

Proof: A generic tangent line is transversal to e and to the edges and vertices of a (P_t, Q_t, R_t)-tuple, by definition and Lemma 3. Moreover, there are $O(p+q+r)$ distinct (P_t, Q_t, R_t)-tuples for t ranging in $[0, \pi]$ by Lemma 7. Since the union of $\{e\}$ and any (P_t, Q_t, R_t)-tuple consists of at most 8 edges or vertices of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$, with at least one of each of the polytopes, the result follows from Lemma 8. $\square$

3 Upper bound for Theorem 2

We prove in this section the upper bound of Theorem 2 on the number of tangents to any 4 among k polytopes. The lower bound is proved in Section 4

Consider k polytopes $\mathbf{P}_1, \ldots, \mathbf{P}_k$, and let n_i denote the number of edges of $\mathbf{P}_i$. Choose an edge e, and let $\mathbf{P}_j$, $\mathbf{P}_l$, and $\mathbf{P}_m$ be distinct polytopes not containing edge e. From the Main Lemma, we know that the number of connected components of tangents to $\mathbf{P}_j$, $\mathbf{P}_l$, and $\mathbf{P}_m$ intersecting e is no more than $C(n_j + n_l + n_m)$, where C is some constant. We sum, over all edges e, the number of tangents intersecting e. There are n edges in the scene, so the number T of connected components of tangents to four polytopes satisfies

$$T \leqslant n \sum_{j<l<m} C\,(n_j + n_l + n_m).$$

Since each n_i, $1 \leqslant i \leqslant k$, appears $\binom{k-1}{2}$ times in the sum, it follows that

$$T \leqslant C\,n \sum_{1 \leqslant i \leqslant k} n_i \binom{k-1}{2} = C\,n^2 \binom{k-1}{2}$$

so T is $O(n^2k^2)$ as claimed.

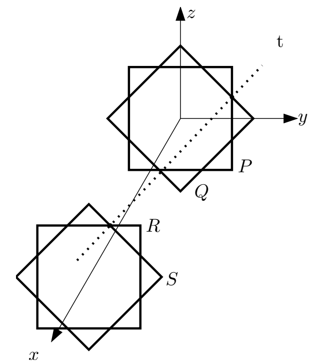

Figure 7: Lower bound example for Lemma 10.

4 Lower bounds for Theorems 1 and 2

We provide in this section the lower bounds needed for Theorems 1 and 2.

Lemma 10 *There exist four polytopes of complexity $O(n)$ such that the number of common tangent lines is finite and $\Omega(n^2)$. There also exist two polytopes of complexity $O(n)$ and two polytopes of complexity $O(m)$ such that the number of common tangent lines is finite and $\Omega(mn)$.*

Proof: We consider four planar regular polygons P, Q, R, and S, each with n vertices, embedded in $\mathbb{R}^3$. P is centered at the origin and parallel to the yz-plane, Q is obtained from P by a rotation of angle $\frac{\pi}{n}$ about the x-axis, and R and S are obtained from P and Q, respectively, by a translation of length 1 in the positive x-direction (see Figure 7). We transform the polygons P and Q into the polytopes $\mathbf{P}$ and $\mathbf{Q}$ by adding a vertex at coordinates $(\varepsilon, 0, 0)$. Similarly, we transform the polygons R and S into the polytopes $\mathbf{R}$ and $\mathbf{S}$ by adding a vertex at coordinates $(1+\varepsilon, 0, 0)$.

For ε sufficiently small, the lines tangent to $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ are the lines through a vertex

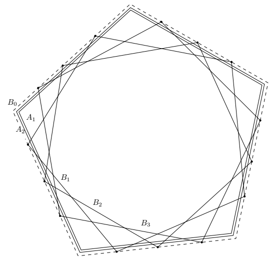

Figure 8: The polygons A_i and B_i.

of $P \cap Q$ and a vertex of $R \cap S$. Since $P \cap Q$ and $R \cap S$ have $2n$ vertices each, there are $4n^2$ tangent lines.

Replacing R and S in the above construction by regular polygons each with m vertices yields the $\Omega(mn)$ lower bound in the case where two polytopes of complexity $O(n)$ and two polytopes of complexity $O(m)$. $\square$

We now provide the lower bound needed for Theorem 2.

Lemma 11 *There exist k disjoint polytopes of total complexity $O(n)$ such that the number of non-occluded line segments tangent to four of them is finite and $\Omega(k^2n^2)$.*

Proof: The lower bound example is similar to the one with four polyhedra. For simplicity suppose that n and k are such that $\frac{n}{k}$ and $\frac{k}{4}$ are integers. We first take a $\frac{n}{k}$-regular polygon A_1 in the plane $x = 0$. Next we consider a copy, B_0, of A_1 scaled by a factor of $(1+\varepsilon)$, and on each edge of B_0 we place $\frac{k}{4}$ points. Polygon B_i, $1 \leqslant i \leqslant \frac{k}{4}$, is constructed by taking the i^{th} point on each edge of B_0. If ε is small enough, the intersection points of A_1 and B_i are outside the other polygons B_j for $1 \leqslant j \leqslant \frac{k}{4}$ and $i \neq j$. Now the A_i, for $2 \leqslant i \leqslant \frac{k}{4}$, are constructed as copies of A_1 scaled by a factor $1 + \frac{i}{k}\varepsilon$ (see Figure 8). For the moment, all polygons lie in plane $x = 0$. We now construct 4 families of $\frac{k}{4}$ polygons each:

- P_i is a copy of A_i translated by $i\epsilon$ in the negative x direction
- Q_i is a copy of B_i translated by $i\epsilon$ in the positive x direction
- R_i is a copy of B_i translated by $1 - i\epsilon$ in the positive x direction
- S_i is a copy of A_i translated by $1 + i\epsilon$ in the positive x direction

Any choice of four polygons, one in each family P_i, Q_j, R_l and S_m, reproduces the quadratic example of Lemma 10 with polygons of size $\frac{n}{k}$ and thus with total number of tangents larger than $\left(\frac{k}{4}\right)^4 4\left(\frac{n}{k}\right)^2 = \frac{n^2k^2}{4}$. Furthermore the lines tangent to P_i, Q_j, R_l and S_m are only occluded by $P_{i'}$ and $S_{m'}$ for $i' > i$ and $m' > m$, that is, beyond the portion of the tangents containing the contact points. The k polygons can be transformed into k convex polyhedra as in Lemma 10. $\square$

References

[1] P. K. Agarwal. On stabbing lines for convex polyhedra in 3D. *Comput. Geom.: Theory and Appl.*, 4(4):177–189, 1994.

[2] M. de Berg, D. Halperin, M. Overmars, and M. van Kreveld, Sparse arrangements and the number of views of polyhedral scenes, In *Internat. J. Comput. Geom. Appl.* 7(3):175–195, 1997.

[3] H. Brönnimann, O. Devillers, V. Dujmovic, H. Everett, M. Glisse, X. Goaoc, S. Lazard, H.-S. Na, and S. Whitesides, On the Number of Lines Tangent to Four Convex Polyhedra. In *Proc. Can. Conf. Comp. Geom.*, pp. 113–117, 2002.

[4] H. Brönnimann, H. Everett, S. Lazard, F. Sottile, and S. Whitesides. The number of transversals to line segments in $\mathbb{R}^3$. In *Proc. Can. Conf. Comp. Geom.*, Halifax, 2003. Also *INRIA Technical report No. 4864*, July 2003.

[5] O. Devillers, V. Dujmovic, H. Everett, X. Goaoc, S. Lazard, H.-S. Na, and S. Petitjean. The expected number of 3D visibility events is linear. *SIAM J. Comput.*, 32(6):1586-1620, June 2003.

[6] F. Durand. *3D Visibility: Analytical study and Applications.* PhD Thesis, Université Joseph Fourier, Grenoble, France, July 1999.

[7] F. Durand. A multidisciplinary survey of visibility. *ACM Siggraph Course Notes: Visibility, Problems, Techniques, and Applications*, 2000.

[8] F. Durand, G. Drettakis, and C. Puech. The visibility skeleton: a powerful and efficient multi-purpose global visibility tool. In *Comput. Graphics Proc. Ann. Conf. Series (SIGGRAPH)* 31:89–100, 1997.

[9] F. Durand, G. Drettakis, and C. Puech. The 3D visibility complex. *ACM Trans. on Graphics*, 21(2):176–206, 2002.

[10] M. Pocchiola and G. Vegter. The visibility complex. *Internat. J. Comput. Geom. Appl.*, 6(3):279–308, 1996.

[11] R. Schiffenbauer. *A Survey of Aspect Graphs.* Technical Report TR-CIS-2001-01, Dept. of Comp. and Info. Science, Polytechnic U., 2001.

A Proofs of Claims A and E of Lemma 5

We give here a complete proof a Claims A and E of Lemma 5 when the polytopes are in arbitrary position. We recall here these two claims for simplicity.

Claim A *Each edge and vertex of u intersects Π_t in exactly one point (possibly on l_e), for all $t \in \mathcal{N}$.*

Moreover, the union of the edges and vertices of u intersects Π_t in at least two distinct points for all $t \in \mathcal{N}$ if u is a maximal (P_{t^*}, Q_{t^*})-tuple, and for all $t \in \mathcal{N}^\star$ otherwise.

Proof of Claim A: We have assumed that u contains no vertex v, or edge with endpoint v, such that v lies on $\Pi_{t^*} \backslash l_e$. It follows that each edge and vertex of u intersects Π_t in at least one point for all $t \in \mathcal{N}$. Moreover, each edge of u either lies in l_e or intersects Π_t in exactly one point, for all $t \in \mathcal{N}$. However, no edge of u lies in l_e because otherwise, if an edge x of say $\mathbf{P}$ belongs to u, then any line tangent to P_t in Π_t and intersecting x contains an endpoint of x which is a vertex of $\mathbf{P}$; thus, by definition, u does not contain x but one of its endpoints. Hence each edge and vertex of u intersects Π_t in exactly one point for all $t \in \mathcal{N}$.

We now prove that the union of the edges and vertices of u intersects Π_t in at least two distinct points for any $t \in \mathcal{N}$ such that u is a maximal (P_t, Q_t)-tuple. Suppose for a contradiction that the union of the edges and vertices of u intersects Π_t in one single point v for such a t. Polygons P_t and Q_t are not both equal to v because otherwise v is a vertex of $\mathbf{P}$ and $\mathbf{Q}$ which lies on l_e (by assumption on u) and thus u is the set of supports of l_e, contradicting the hypotheses. Now, since u is a (P_t, Q_t)-tuple, there exists a line in Π_t tangent to P_t and Q_t whose set of supports is u. This line can be rotated about v until it gets tangent to P_t or Q_t in some other point. Hence u is not a maximal (P_t, Q_t)-tuple, contradicting our hypotheses.

Thus, if u is a maximal (P_t, Q_t)-tuple for all $t \in (t^*, t_1)$, the union of the edges and vertices of u intersects Π_t in at least two distinct points for all $t \in (t^*, t_1)$ and thus for all $t \in \mathcal{N}^\star$. Also, if u is a maximal (P_t, Q_t)-tuple for $t = t^*$, the union of the edges and vertices of u intersects Π_t in at least two distinct points for $t = t^*$ and thus for all $t \in \mathcal{N}$. □

Claim E *Line m is tangent to $\mathbf{Q} \cap \Psi$ at some point not on l_e.*

Proof of Claim E: By Claim C, m is tangent to Q_{t^*} and thus to $\mathbf{Q}$. It follows that m is tangent to $\mathbf{Q} \cap \Psi$ or properly intersects it, in which case $\mathbf{Q} \cap \Psi$ is a face or an edge of $\mathbf{Q}$. Suppose for a contradiction that m does not satisfy the claim. It means that m either is tangent to $\mathbf{Q} \cap \Psi$ on l_e (see Figure 9(a)), or is tangent to $\mathbf{Q}$ along a face or an edge in Ψ and properly intersects it (see Figure 9(b)). Note that these situations are not generic since two edges, one in $\mathbf{Q}$ and one in $\mathbf{P}$ or e, are then coplanar.

First, by Claims C and D, $\Psi \cap \Pi_{t^*}$ is a line m, thus plane Π_t intersects Ψ in a line for all $t \in \mathcal{N}$. We denote that line by m_t. We prove that u is the set of supports of m_t for all $t \in \mathcal{N}$, and conclude that u is a maximal (P_t, Q_t)-tuple for all $t \in \mathcal{N}^*$, and derive a contradiction with the hypothesis of Lemma 5.

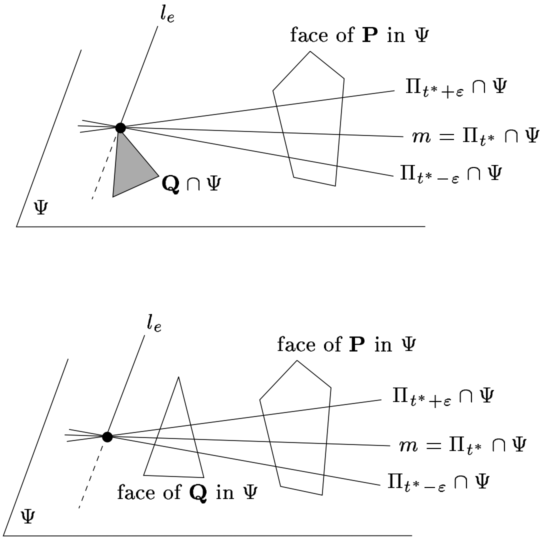

Figure 9: m is tangent to $\mathbf{P}$ along a face in Ψ and (a) to $\mathbf{Q} \cap \Psi$ only on l_e or (b) to $\mathbf{Q}$ along a face in Ψ.

We prove first that u is contained in the set of supports of m_t for all $t \in \mathcal{N}$. By Claim C, the set of supports of m contains u, so it is sufficient to prove that m_t intersects the supports of m that belong to u, for any $t \in \mathcal{N}$. Since m is assumed not to satisfy the claim, a support edge of m that belongs to $\mathbf{P}$ or $\mathbf{Q}$ either meets m on l_e or lies in the plane Ψ. In both cases, m_t intersects it for all $t \in \mathcal{N}$ because m contains no endpoint of its support edge (by definition). A support vertex of m that lies on l_e belongs to Ψ and Π_t for every t and thus belongs to m_t for all $t \in \mathcal{N}$. Finally, a support vertex of m that does not lie on l_e does not belong to u (by assumption on u). Hence, for all $t \in \mathcal{N}$ the line m_t intersects all the supports of m that belong to u, thus the supports of m_t contain u.

By Claim A, the union of the edges and vertices of u intersects Π_t in at least two distinct points, thus m_t is the only line in Π_t whose set of supports can be u. It follows that the set of supports of m_t is u for any t such that u is a (P_t, Q_t)-tuple, that is for $t = t^*$ or all $t \in (t^*, t_1)$.

Recall that u contains no vertex v, or edge with endpoint v, such that v lies on $\Pi_{t^*} \setminus l_e$. Thus, for $t = t^*$ or all $t \in (t^*, t_1)$, m_t contains no vertex v and intersects no edge with endpoint v such that v lies in $\Pi_t \setminus l_e$. It follows, either directly if u is a (P_{t^*}, Q_{t^*})-tuple or by a continuity argument otherwise, that m contains no vertex v and intersects no edge with endpoint v such that v lies in $\Pi_{t^*} \setminus l_e$. Therefore, the set of supports of m_t is invariant for $t \in \mathcal{N}$ (see Figure 9).

Now, line m_t varies continuously with t, its set of supports is invariant for $t \in \mathcal{N}$, and $m = m_{t^*}$ is tangent to P_{t^*} and Q_{t^*} (by Claim C). Thus m_t is tangent to P_t and Q_t for all $t \in \mathcal{N}$, which yields that u is a (P_t, Q_t)-tuple for all $t \in \mathcal{N}$.

By Claim A, m_t is the only line whose set of supports contains u for $t \in \mathcal{N}^\star$. Thus u is a maximal (P_t, Q_t)-tuple for all $t \in \mathcal{N}^\star$, and the hypothesis of Lemma 5 implies that u is a non-maximal (P_{t^*}, Q_{t^*})-tuple. Thus the union of all the edges and vertices of u intersect Π_{t^*} in one point, which is a contradiction since $m = \Pi_{t^*} \cap \Psi$ is tangent to P_{t^*} along an edge by Claim C. $\square$

B Proof of Lemma 8

We give in this section a complete proof of Lemma 8 which states:

The set of transversals to e and to $O(1)$ edges or vertices of **P**, **Q**, *and* **R**, *with at least one of each of the polytopes, contains at most $O(1)$ connected components of tangents to* **P**, **Q**, **R** *and* **S**.

The set of transversals to $O(1)$ segments (possibly of zero length) consists of $O(1)$ connected components [4]. The result directly follows if the set of transversals is finite. Otherwise, let $\mathcal{T}$ denote the set of transversals.

We prove that $\mathcal{T}^+$, the subset of $\mathcal{T}$ of lines tangent to **P**, **Q**, **R** and **S**, consists of at most $O(1)$ connected components. Let $\mathcal{T}^- = \mathcal{T} \backslash \mathcal{T}^+$ and $\mathcal{T}^\partial$ denote the frontier between $\mathcal{T}^+$ and $\mathcal{T}^-$.

The number of connected components of $\mathcal{T}^+$ is bounded by the sum of the numbers of connected components of $\mathcal{T}$ and $\mathcal{T}^\partial$. As mentioned above, $\mathcal{T}$ consists of $O(1)$ connected components. We show that $\mathcal{T}^\partial$ consist of $O(1)$ connected components.

$\mathcal{T}^\partial$ consists of lines of $\mathcal{T}$ that are tangent to **P**, **Q**, **R** and **S**, and go through a vertex or lie in a plane containing a face of one of these polytopes. In the latter case, we say that the line is tangent to the polytope along a 2d face. We consider in turn cases depending on the dimension d of $\mathcal{T}$ (in line space). d is equal to 1, 2 or 3 because, since $\mathcal{T}$ is not discrete, $d \geqslant 1$ and, since the lines of $\mathcal{T}$ intersect a non-empty set of edges, $d \leqslant 3$.

Case $d = 1$. We prove that $\mathcal{T}^\partial$ consists of at most 20 lines. The lines of $\mathcal{T}$ lie in a surface, Υ, which is a plane, a pair of planes, a hyperbolic paraboloid or a hyperboloid of one sheet [4]. We consider each case in turn.

Suppose first that Υ is a plane. If Υ is tangent to **S**, all the lines of $\mathcal{T}$ are tangent to **S**, and similarly for **P**, **Q** and **R**. Otherwise Υ properly intersects **S** in a convex polygon, ρ. A line of $\mathcal{T}$ is tangent to **S** along a face only if it is tangent to ρ along an edge. Moreover, since the lines of $\mathcal{T}$ intersect the edge e of **S**, they intersect the vertex $e \cap \Upsilon$ of ρ. Thus $\mathcal{T}$ contains at most 2 lines tangent to **S** along a face, and similarly for **P**, **Q** and **R**. Hence $\mathcal{T}^\partial$ consists of at most 8 lines.

If the surface Υ is two planes, then similarly, $\mathcal{T}^\partial$ consists of at most 16 lines.

If the surface Υ is a hyperbolic paraboloid or a hyperboloid of one sheet, then no two lines of $\mathcal{T}$ intersect e at the same point. Thus e lies on Υ because all the lines of $\mathcal{T}$ contain different points of e. Moreover, only 2 lines of $\mathcal{T}$ intersect an endpoint of e. These 2 lines might be tangent to **S** along a face. Any other line tangent to **S** along a face intersects the relative interior of e and thus is tangent to **S** along one of the 2 faces of **S** incident to e. The line thus belongs to the intersection of Υ with one of the 2 planes supporting these faces. The intersection of Υ and each of these planes consists of the line containing e and another line because both Υ and the plane contain e and Υ is a quadric. Thus $\mathcal{T}$ contains at most 5 lines tangent to **S** along a face, and similarly for **P**, **Q** and **R**. Hence $\mathcal{T}^\partial$ consists of at most 20 lines.

Case $d = 2$. $\mathcal{T}$ is a set of lines intersecting 2 non-overlapping line segments or at least 3 non-overlapping concurrent line segments.

In the former case, if the 2 line segments are coplanar in a plane Υ, the lines of $\mathcal{T}$ intersecting the edge e are tangent to **S** along a face only if the face lies in Υ. Then all the lines of $\mathcal{T}$ are tangent to **S** along this face and thus $\mathcal{T}^\partial$ is empty. Similarly for **P**, **Q** and **R**.

Consider now the case where $\mathcal{T}$ is a set of lines intersecting 2 skew line segments. Edge e contains one of them; assume for simplicity that edge e is one of them and let e' denote the other one. A line of $\mathcal{T}$ intersecting e and tangent to **S** along a face is either contained in one of the 2 planes containing a face of **S** incident to e, or goes through an endpoint of e. If the plane containing a face of **S** incident to e intersects the edge e', then the intersection is a point (because e and e' are skew) and all the lines through it and intersecting e are tangent to **S** along a face. Thus $\mathcal{T}$ contains at most one connected component of lines tangent to **S** along each of the 2 faces incident to e. Now, the set Ξ of lines through one endpoint of e and intersecting the other edge e' defines a plane. Either this plane contains a face of **S** and all the lines of Ξ are tangent to **S** along a face, or not and at most one line of Ξ is tangent to **S** along a face. Thus $\mathcal{T}$ contains at most 4 connected components of lines tangent to **S** along a face, and similarly for **P**, **Q** and **R**. Hence $\mathcal{T}^\partial$ consists of at most 16 connected components of lines.

Consider now the latter case where $\mathcal{T}$ is a set of lines intersecting at least 3 non-overlapping concurrent line segments. If these lines are coplanar then, as before, $\mathcal{T}^\partial$ is empty. Otherwise, $\mathcal{T}$ is a set of lines going through a point. This point lies on e, thus all the lines of $\mathcal{T}$ tangent to **S** along a face form one connected component, and similarly for **P**, **Q** and **R**. Hence $\mathcal{T}^\partial$ consists of at most 4 connected components of lines.

Case $d = 3$. $\mathcal{T}$ is a set of lines intersecting 1 line segment. Edge e contains this line segment thus all the lines of $\mathcal{T}$ tangent to **S** along a face form one connected component, and similarly for **P**, **Q** and **R**. Hence $\mathcal{T}^\partial$ consists of at most 4 connected components of lines.

We proved for all possible dimensions of $\mathcal{T}$ that $\mathcal{T}^\partial$ and thus $\mathcal{T}^+$ consists of $O(1)$ connected components of lines. Thus $\mathcal{E}$, which contains all the generic tangent lines, contains $O(p + q + r)$ connected components of lines intersecting e and tangent to **P**, **Q**, **R** and **S**.

C Nongeneric tangent lines

In this section we count the number of nongeneric tangent lines. Note that, as mentioned before, there are no such lines under some adequate general position assumption.

Lemma 12 *There are at most $O(p + q + r)$ connected components of nongeneric tangent lines.*

We first introduce some notation. A sweep plane is called a *tangent plane* if and only if it is tangent to **P**, **Q**, or **R** and at least one point of tangency does not lie on l_e. We prove three lemmas which yields Lemma 12.

Lemma 13 *A nongeneric tangent line lies in a tangent plane or properly intersects an edge of* **P**, **Q**, *or* **R** *lying in* l_e.

Proof: Let l be a nongeneric tangent line. By definition, l properly intersects P_t, Q_t, or R_t in some plane Π_t. Suppose without loss of generality that l properly intersects P_t. Since l is tangent to **P**, P_t is a face or an edge of **P** which implies that Π_t is tangent to **P**. Thus Π_t is a tangent plane or P_t is an edge contained in l_e. In the first case, l lies in the tangent plane Π_t. In the latter case, l properly intersects an edge of **P** lying in l_e. □

We count first the nongeneric tangent lines lying in tangent planes and then the one that properly intersect an edge of **P**, **Q**, or **R** lying in l_e.

Lemma 14 *The nongeneric tangent lines lying in all tangent planes form* $O(p+q+r)$ *connected components.*

Proof: We first consider the nongeneric tangent lines lying in a given tangent plane Π_t. Such lines intersect, properly or not, e, P_t, Q_t, and R_t in the plane Π_t. Let l denote such a line and $\mathcal{A}_p$ (resp. $\mathcal{A}_t$) the subset of $\{e, P_t, Q_t, R_t\}$ of polygons that properly intersect l (resp. are tangent to l). We can move l in Π_t into a line tangent to at least two polygons of $\{e, P_t, Q_t, R_t\}$, while intersecting (properly ot not) the polygons of $\mathcal{A}_p$ and remaining tangent to the polygons of $\mathcal{A}_t$ during the motion. There are $O(p+q+r)$ such tangents, thus the nongeneric tangent lines lying in Π_t form $O(p+q+r)$ connected components.

The result follows since there are at most 2 tangent planes per polytope. □

Lemma 15 *The nongeneric tangent lines that properly intersect an edge of* **P**, **Q**, *or* **R** *lying in* l_e. *form* $O(p+q+r)$ *connected components.*

Proof: Let l denote such a line. Let o_p be the union of edges of **P** that are contained in l_e; if no such edge exists, o_p is empty. We define o_q and o_r similarly. At least one of o_p, o_q, and o_r is nonempty, l intersects $\alpha = 1$, 2, or 3 of o_p, o_q, and o_r. We consider in turn the three cases depending on the value of α.

Case $\boldsymbol{\alpha} = 3$. Sets o_p, o_q, and o_r are all nonempty since l intersects each of them. If o_p, o_q, o_r and e do not all pairwise intersect, i.e., if $I = e \cap o_p \cap o_q \cap o_r$ is empty then there is exactly one line l stabbing all four o_p, o_q, o_r and e, and that is line $l = l_e$. If on the contrary, I is nonempty, then the nongeneric tangent lines intersecting I are the line intersecting I and not intersecting the interior of **P**, **Q**, and **R**. These lines thus form one connected component.

Case $\boldsymbol{\alpha} = 2$. Suppose that l intersects o_p and o_q; the other cases are similar. Then o_p and o_q are nonempty. Again if $I = e \cap o_p \cap o_q$ is empty, then there is at most one line $l = l_e$ intersecting e, o_p, and o_q and tangent to **R**. If on the contrary, I is nonempty then there are at most r edges of **R** that may induce some tangents with I. By Lemma 8, that implies that there are at most $O(r)$ connected components of lines intersecting e, o_p, and o_q, and tangent to **R**.

Case $\boldsymbol{\alpha} = 1$. Suppose that l intersects o_p; the other cases are similar. Then o_p is nonempty. Again if $I = e \cap o_p$ is empty, then there is at most one line $l = l_e$ intersecting both o_p and e and tangent to polytopes **Q** and **R**. Consider now the case where the common intersection I is nonempty. Lemmas 9, 13 and 14, when applied to I and the three polytopes **Q**, **R**, and **S** = **R** yields that there are at most $O(q+r)$ connected components of lines intersecting I and tangent to **Q** and **R** such that none of the points of tangency lie on l_e. □