14

Three-dimensional drawings

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14.1 Introduction

Two dimensional graph drawing, that is, graph drawing in the plane, has been widely studied. While this is not yet the case for graph drawing in 3D, there is nevertheless a growing body of research on this topic, motivated in part by advances in hardware for three-dimensional graphics, by experimental evidence suggesting that displaying a graph in three dimensions has some advantages over 2D displays [WF94, WF96, WM08], and by applications in information visualization [WF94, WM08], VLSI circuit design [LR86], and software engineering [WHF93]. Furthermore, emerging technologies for the nano through micro scale may create demand for 3D layouts whose design criteria depend on, and vary with, these new technologies.

Not surprisingly, the mathematical literature is a source of results that can be regarded as early contributions to graph drawing. For example, a theorem of Steinitz states that a graph G is a skeleton of a convex polyhedron if and only if G is a simple 3-connected planar graph.

It is natural to generalize from drawing graphs in the plane to drawing graphs on other surfaces, such as the torus. Indeed, surface embeddings are the object of a vast amount of research in topological graph theory, with entire books devoted to the topic. We refer the interested reader to the book by Mohar and Thomassen [MT01] as an example.

Numerous drawing styles or conventions for 3D drawings have been studied. These styles differ from one another in the way they represent vertices and edges. We focus on the most common ones.

In this chapter, by a *drawing* we always mean a graph representation (realization, layout, embedding) where no two vertices overlap and no vertex-edge intersections occur unless there is a corresponding vertex-edge incidence in the combinatorial graph. We say that two edges *cross* if they intersect at a point that is not the location of a shared endpoint of the edges in the combinatorial graph. A drawing is *crossing-free* if no two edges cross.

It is natural to represent each vertex by a point and each edge by a straight line segment joining its endpoint vertices. These so-called *straight-line* drawings are one of the earliest

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drawing styles considered both in the plane and in 3D. Steinitz' Theorem, for example, assures the existence of 3D straight-line crossing-free drawings of all 3-connected planar graphs. In fact, as will be seen later, all graphs have such drawings in 3D.

Regardless of the application, the placement of vertices is usually limited to points in some discretized space. For example, when a drawing is to be displayed on a computer screen, vertices must be mapped to integer grid points (pixels). This motivates the study of *grid drawings*, where vertices are required to have integer coordinates. An attractive feature of such drawings is that they assure a minimum separation of at least one grid unit between any pair of vertices. This aids readability and is thus a desirable aesthetic in visualization applications.

Straight line crossing-free drawings whose vertices are located at points in $\mathbb{Z}^3$ are called *3D (straight-line) grid drawings*. The relaxation where edges are represented with polygonal chains with bends (if any) also at grid-points gives rise to the so-called *3D polyline grid drawings*. Here, a point where a polygonal chain changes its direction is called a *bend*. Straight-line grid drawings are thus a special case of polyline grid drawings. Polyline drawings provide great flexibility. In particular, they allow 3D drawings with smaller volume than is possible in the straight-line model. The number of bends, however, should be kept as small as possible, since bends typically reduce the readability of a drawing.

If each segment of each edge in a polyline drawing is parallel to one of the three coordinate axes, then we say the drawing is an *orthogonal drawing*. Orthogonal drawings are thus special cases of polyline drawings. Since the orthogonal style guarantees very good angular resolution, it is commonly chosen for VLSI design and data-flow diagrams. However, since each vertex is represented by a point, for a graph to admit a 3D orthogonal drawing, each vertex must have degree at most six. To overcome this difficulty, *orthogonal box drawings* were introduced, where each vertex is represented by an axis-aligned box. In such drawings, in addition to the volume and number of bends, various aspects of the sizes and shapes of the boxes are taken as quality measures for the drawing.

Different drawing styles may be subject to different measures of quality. More often than not, however, the measure of a good drawing, regardless of its purpose, rewards having few edge crossings. When a drawing is to be displayed on a page or a computer screen, or is to be used for VLSI design, it is important to keep the volume small to avoid wasting space. On the other hand, a bend on an edge increases the difficulty for the eye to follow the course of the edge. For this reason, it is desirable to keep the edges straight, or at least to keep small the total number of bends and the maximum number of bends per edge.

Since by definition 3D grid drawings have straight edges and no crossings, volume is the main aesthetic criterion for this drawing style. The convention for measuring the volume of a drawing is to multiply together the number of grid points on each of three mutually orthogonal sides of the axis-aligned bounding box of the drawing. In polyline and orthogonal 3D drawings, in addition to the volume, the number of bends is a measure of the quality of the drawing.

In the last decade, this topic has been extensively studied by the graph drawing community. Hence much of the following chapter, in particular Sections 14.2 and 14.3, is dedicated to reviewing the results obtained for 3D (polyline) grid drawings and 3D orthogonal drawings with the volume and the number of bends as the main aesthetic criteria.

Other measures of quality for 3D drawings include: *angular resolution*, defined as the size of the smallest angle between any pair of edges incident to the same vertex; *aspect ratio*, which is the ratio of the length of the longest side to the length of the shortest side of the bounding box of the drawing; and *edge resolution*, which is the minimum distance between a pair of edges not incident to the same vertex. When the underlying combinatorial graph has non-trivial automorphisms, displaying some of the symmetries of the graph can produce beautiful drawings. The display of symmetry in a 3D drawing is the topic of Section 14.5.

Suppose edge crossings are permitted for graphs drawn in the plane, but that the edges must then be colored so that no two edges that cross each other have the same color. The minimum number of colors, taken over all possible drawings of that graph, is the classical graph parameter known as *thickness*. If the edges are required to be straight, then this parameter is called the *geometric thickness*. If, in addition, the vertices are required to lie in convex position (i.e., the convex hull of the vertices contains no vertices in its interior), then the parameter is called the *book thickness*.

These three extensively studied graph parameters have a natural interpretation in 3D graph drawing that is important for multilayered VLSI design. Undesired crossings of uninsulated wires are avoided by having wires placed onto several different physical layers, making each layer crossing-free. The graph drawing convention associated with this application area represents each vertex as a line-segment parallel to the Z-axis. Each vertex is intersected by all layers (that is, by planes orthogonal to the Z-axis). Each edge is confined to one of the layers and is drawn between its endpoints in its layer. Edges in the same layer are not allowed to cross. Associating layers, and the edges placed in them, with colors, clearly two edges with the same color do not cross. Thus the minimum possible number of layers corresponds to the thickness parameter. Motivated by the fact that only a limited but increasing number of layers is possible in VLSI technology and also noting that a small number of layers is easier for humans to understand visually, the number of layers of a drawing, that is, its thickness, is the main criterion for the quality for such drawings. The thickness parameters are the subject of Section 14.4.

Graph theory notation used in this chapter: In what follows, all graphs are simple unless stated otherwise. A multigraph is a graph with no loops but it may have multiple copies of edges. A graph G with $n = |V(G)|$ vertices, $m = |E(G)|$ edges, maximum degree at most Δ, and chromatic number c is referred to as an n-vertex m-edge degree-Δ c-colorable graph. The complete graph on n vertices is denoted by K_n.

A graph H is a minor of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A class of graphs is *minor-closed* if for any graph in the class, all its minors are also in the class. For example, the class of all planar graphs is minor-closed since contracting and/or deleting an edge in a planar graph results in another planar graph. On the contrary, contracting an edge in a 4-regular graph may result in a vertex of degree higher than 4, thus the class of all 4-regular graphs is not minor-closed. A minor-closed class of graphs is *proper* if it is not the class of all graphs.

14.2 Straight-line and polyline grid drawings

14.2.1 Straight-line grid drawings

A *three-dimensional straight-line grid drawing*[1] of a graph, henceforth called a *3D grid drawing*, represents the vertices by distinct points in $\mathbb{Z}^3$ (called *grid-points*), and represents each edge as a line-segment between its endpoints, such that edges only intersect at common endpoints, and an edge intersects only the two vertices that are its endpoints. In contrast to the case for the plane, every graph has a 3D grid drawing, by a folklore construction. It is therefore of interest to optimize certain quality measures of such drawings. The most

[1]sometimes called a *three-dimensional Fary grid drawing*

commonly studied measure for 3D grid drawings is their volume, measured as follows.

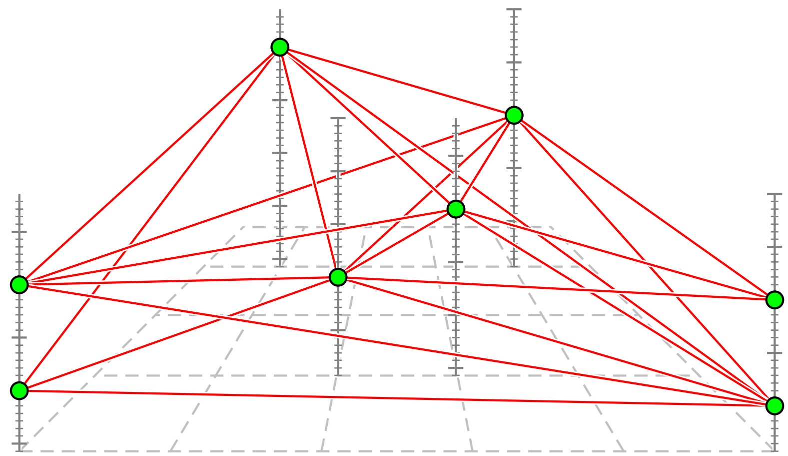

Figure 14.1 A 3D grid drawing of a graph.

The *bounding box* of a 3D grid drawing is the minimum axis-aligned box containing the drawing. If the bounding box has side lengths $X-1$, $Y-1$ and $Z-1$, then we speak of an $X \times Y \times Z$ grid drawing with *volume* $X \cdot Y \cdot Z$. That is, the volume of a 3D grid drawing is the number of gridpoints in the bounding box. This definition is formulated so that two-dimensional straight-line grid drawings have positive volume.

A starting point for many results on 3D grid drawings is the following simple fact.

Fact 14.1 *A straight-line drawing of a graph (on $n > 3$ vertices) such that no four vertices are coplanar has no crossings.*

This fact is key to the folklore construction that proves that every graph has a 3D grid drawing. In particular, a *moment curve* M is a curve defined by parameters (q, q^2, q^3). It is not difficult to prove that no four distinct points on this curve are coplanar. Thus given a graph G on n vertices, a 3D grid drawing of G can be obtained by placing each vertex $v_i \in V(G)$, $1 \le i \le n$, at (i, i^2, i^3). This construction gives an $n \times n^2 \times n^3$ 3D grid drawing with $\mathcal{O}(n^6)$ volume. Cohen *et al.* [CELR96] improved this bound by placing each vertex v_i at the grid-point $(i, i^2 \bmod p, i^3 \bmod p)$, where p is a prime with $n < p \le 2n$. The resulting drawing is an $n \times 2n \times 2n$ 3D grid drawing with $\mathcal{O}(n^3)$ volume. This construction is a generalization of an analogous two-dimensional technique due to Erdös [Erd51]. Furthermore, Cohen *et al.* [CELR96] proved that the $\Omega(n) \times \Omega(n) \times \Omega(n)$ bounding box and thus the $\Theta(n^3)$ volume bound is asymptotically optimal in the case of the complete graph K_n. The proof of this lower bound is based on the fact that in any 3D grid drawing of K_n, no five vertices can be coplanar, so each side of the bounding box has size at least $n/4$.

Theorem 14.1 [CELR96] *Every n-vertex graph has a 3D grid drawing with $\mathcal{O}(n^3)$ volume. Moreover, the bounding box of every 3D grid drawing of K_n, the complete graph on n vertices, is at least $\frac{n}{4} \times \frac{n}{4} \times \frac{n}{4}$, and thus has $\Omega(n^3)$ volume.*

Since complete graphs require cubic volume, it is of interest to identify fixed graph parameters that allow for 3D grid drawings with smaller volume. The first such parameter to be studied was the chromatic number [CS97, PTT99]. Calamoneri and Sterbini [CS97] proved that each 4-colorable graph has a 3D grid drawing with $\mathcal{O}(n^2)$ volume. Generalizing this result, Pach *et al.* [PTT99] proved the following theorem.

Theorem 14.2 [PTT99] *Every n-vertex graph with chromatic number χ has a 3D grid drawing with $\mathcal{O}(\chi^2 n^2)$ volume. This bound is asymptotically optimal for the complete bipartite graphs with equal sized bipartitions.*

The main idea behind this result is similar to the one for general graphs. In case of complete graphs, crossings are avoided by ensuring that no four vertices are coplanar. That restriction, however, necessarily leads to cubic volume 3D grid drawings and is overly cautious for graphs that have small chromatic number. In particular, vertices that belong to the same color class may all be coplanar, as there are no edges between them. To avoid crossings, it suffices to ensure that if two edges share an endpoint, that they are not collinear and otherwise, that they are not coplanar. The construction in [PTT99] does exactly that. All the vertices that belong to the same color class have the same x-coordinate; in particular, they all belong to some plane orthogonal to the X-axis. Edge crossings are then avoided by appropriate choice of y- and z-coordinates for the vertices. Specifically, if $p \in \mathcal{O}(n)$ is a suitably chosen prime, the main step of this algorithm represents the vertices in the i-th color class by grid-points in the set $\{(i, t, it) : t \equiv i^2 \pmod{p}\}$. It follows that the volume bound is $\mathcal{O}(c^2 n^2)$ for c-colorable graphs.

Many interesting graph families have bounded chromatic number, including planar graphs, bounded genus graphs, and bounded treewidth graphs. In fact all proper minor-closed families have bounded chromatic number. By the above result, all such families have 3D grid drawings with quadratic volume. This naturally gives rise to the question of which graph families admit 3D grid drawings with subquadratic, or even linear volume for each member of a class. Since n distinct points on the 3D integer grid cannot fit in a sublinear volume bounding box, linear volume grid drawings are the best possible for any graph. Pach *et al.* [PTT99] proved that the quadratic volume bound is asymptotically optimal for the complete bipartite graph with equal sized bipartitions. This was generalized by Bose *et al.* [BCMW04] for all graphs.

Theorem 14.3 [BCMW04] *Every 3D grid drawing with n vertices and m edges has volume at least $\frac{1}{8}(n+m)$. In particular, the maximum number of edges in an $X \times Y \times Z$ drawing is exactly $(2X-1)(2Y-1)(2Z-1) - XYZ$.*

For example, graphs admitting 3D grid drawings with $\mathcal{O}(n)$ volume have $\mathcal{O}(n)$ edges.

Planar graphs are one natural class to consider as a candidate for admitting 3D grid drawings with small volume. They have chromatic number at most four, and thus, by the above results [CS97][PTT99], they admit $\mathcal{O}(n^2)$ volume 3D grid drawings. More strongly, the classical result of de Fraysseix *et al.* [dFPP90] and Schnyder [Sch89] states that every planar graph has a $1 \times \mathcal{O}(n) \times \mathcal{O}(n)$ 3D grid drawing, that is, planar graphs admit 2D grid drawings in $\mathcal{O}(n^2)$ area. In 2D this is the best possible, as there are planar graphs that require quadratic area. Intuition suggests, however, that in 3D one should be able to do better. The following open problem has been first suggested by Felsner *et al.* [FLW01].

Open Problem 14.1 [FLW01] *Do planar graphs admit linear volume 3D grid drawings?*

Although the problem is still open, in a recent breakthrough, Di Battista *et al.* [BFP10] showed that planar graphs admit $\mathcal{O}(n \log^{16} n)$ volume 3D grid drawings. Some progress has also been made for more general classes of graphs. In particular, all proper minor-closed families of graphs have been proved to admit $\mathcal{O}(n^{\frac{3}{2}})$ volume 3D grid drawings [DW04c]. Refer to Table 14.1 for exact bounds.

Most, if not all, of the successful attempts to derive linear volume bounds have been done by constructing 3D grid drawings that fit in a bounding box with dimensions $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$. In such a drawing all the vertices lie on $\mathcal{O}(1)$ parallel lines. Thus not only does such a drawing have many quadruples of vertices that are coplanar, but in fact a constant fraction of all vertices are collinear.

Consider a drawing of a graph where all vertices lie on t lines parallel to the Z-axis, such that no 3 lines are coplanar and no two vertices on the same line are adjacent. Suppose there is a pair of edges that cross in such a drawing and that we would like to remove just that one crossing. If the four endpoints of the edges belong to four distinct parallel lines, as illustrated in Figure 14.2, then, for example, increasing the z-coordinate of the highest vertex removes the crossing. Whenever four endpoints belong to three distinct lines, the two edges do not cross in the projection to the XY-plane and thus cannot cross in the drawing. If, however, the endpoints belong to two parallel lines, then the only way to remove the crossing is to change the ordering of the vertices on one of the two lines, as illustrated in Figure 14.2. These are the difficult crossings to handle, as they arise from a combinatorial situation of "bad" vertex orderings. Having that in mind, Dujmović *et al.* [DMW02] introduced track layouts of graphs, although similar structures are implicit in much previous work [FLW01, HLR92, HR92, RVM95].

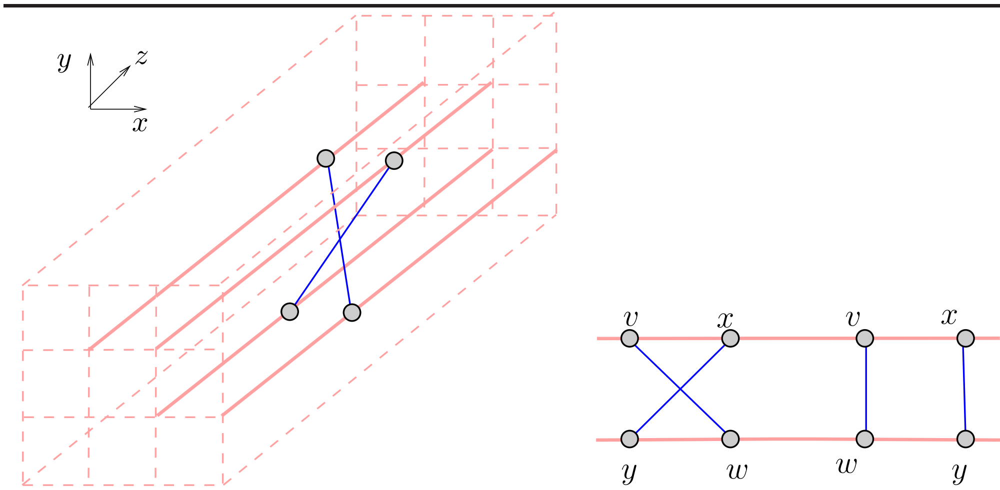

Figure 14.2

Let $\{V_i : i \in I\}$ be a proper vertex t-coloring of a graph G. Let $<_i$ be a total order on each color class V_i. Then $\{(V_i, <_i) : i \in I\}$ is a *t-track assignment* of G. An *X-crossing* in a track assignment consists of two edges vw and xy such that $v <_i x$ and $y <_j w$, for distinct colors i and j. A *t-track layout* of G is a t-track assignment of G with no X-crossing. The *track-number* of G, denoted by $\mathsf{tn}(G)$, is the minimum integer t such that G has a t-track layout. Some authors [dGLMW05, dG03, dGLW02, dGM03] use a slightly different definition of track layout (called *improper*), in which *intra-track* edges are allowed between consecutive vertices in a track.

Track layouts, which are a purely combinatorial structure, and 3D grid drawings are intrinsically related. In particular, a graph G has a $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ 3D grid drawing if and only if G has $\mathcal{O}(1)$ track number [DMW05]. More precisely:

Theorem 14.4 [DMW05, DW04c] *Let G be an n-vertex graph with chromatic number $\chi(G) = c$ and track-number $\mathsf{tn}(G) = t$. Then:*

(a) G has an $\mathcal{O}(t) \times \mathcal{O}(t) \times \mathcal{O}(n)$ 3D grid drawing with $\mathcal{O}(t^2 n)$ volume, and
(b) G has an $\mathcal{O}(c) \times \mathcal{O}(c^2 t) \times \mathcal{O}(c^4 n)$ 3D grid drawing with $\mathcal{O}(c^7 tn)$ volume.

Conversely, if a graph G has an $X \times Y \times Z$ 3D grid drawing, then G has track-number $\mathsf{tn}(G) \leq 2XY$.

The key to proving part (a) of the theorem is knowing that there are no bad orderings, that is, no X-crossings; the rest is a generalization of the number theoretic teachings of Erdös that assigns appropriate z-coordinates to vertices such that crossings between edges whose endpoints belong to four distinct tracks are avoided. Proving part (b) of this theorem is much more involved.

Theorem 14.4 (a) says that graphs that have bounded track number admit linear volume 3D grid drawings. Part (b) says that graphs that have bounded chromatic number and sub-linear track number have sub-quadratic 3D grid drawings. This provides a strong motivation for studying track layouts of different graph families. Consider first a few simple examples. A *caterpillar* is a tree such that deleting the leaves gives a path. It is simple to verify that a graph has track-number two if and only if it is a caterpillar. Trees have track number at most three. That can be verified by starting with a natural 2D crossing-free drawing of a tree, then wrapping it around a triangular prism, as illustrated in Figure 14.3.

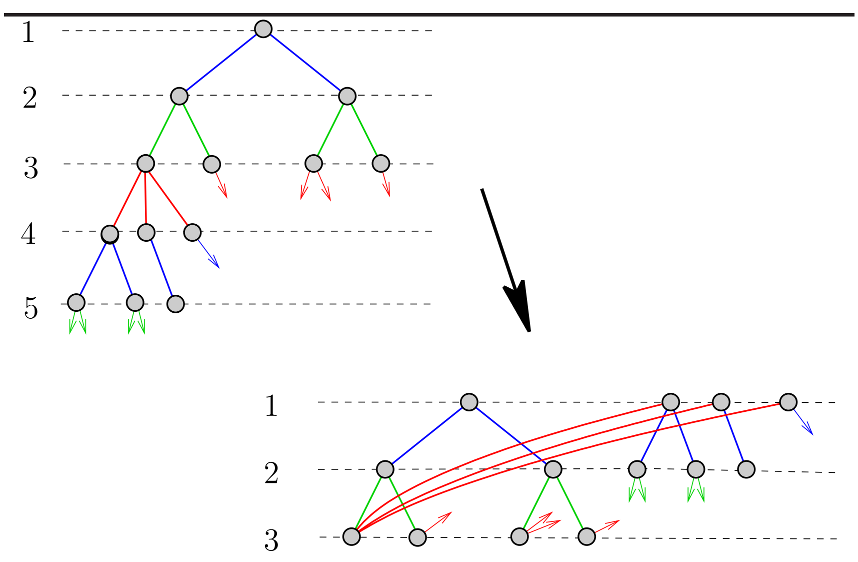

Figure 14.3 3-track layout of trees.

For track layouts such that no two adjacent vertices are allowed to be in the same track, the chromatic number of a graph is a lower bound for its track number. For example, $\mathsf{tn}(K_n) = n$. However, that lower bound is very weak. Observe, for example, that the complete bipartite graph $K_{n,n}$, although 2-colorable, has track number $n+1$: if two vertices from the same bipartition belong to the same track, then no pair of vertices from the other bipartition can lie on the same track, as otherwise that would imply that $K_{4,4}$ has track number two.

The concept of track layouts, in the case of three tracks, is implicit in the work of Felsner *et al.* [FLW01]. They established the first non-trivial $\mathcal{O}(n)$ volume bound for outerplanar graphs. Their algorithm "wraps" a two-dimensional drawing around a triangular prism. They proved that outerplanar graphs have improper track number at most three.

Dujmović *et al.* [DMW05] proved that graphs of bounded treewidth have bounded track number and therefore have linear volume 3D grid drawings. Many graphs arising in applications of graph drawing have small tree-width. Outerplanar and series-parallel graphs are the obvious examples. They have treewidth at most two. Another example arises in software engineering applications. Thorup [Tho98] proved that the control-flow graphs of go-to free programs in many programming languages have treewidth bounded by a small constant: in particular, 3 for Pascal and 6 for C. Other families of graphs having bounded tree-width (for constant k) include: almost trees with parameter k, graphs with a feedback vertex set of size k, band-width k graphs, cut-width k graphs, planar graphs of radius k, and k-outerplanar graphs. If the size of a maximum clique is a constant k then chordal, interval and circular arc graphs also have bounded tree-width.

Note that bounded tree-width is not necessary for a graph to have a 3D grid drawing with $\mathcal{O}(n)$ volume. The $\sqrt{n} \times \sqrt{n}$ plane grid graph has $\Theta(\sqrt{n})$ tree-width and has a $\sqrt{n} \times \sqrt{n} \times 1$ grid drawing with n volume. It also has a 3-track layout (simply wrap the grid graph, along its diagonals, around a triangular prism,) and thus has a $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ 3D grid drawing.

The track number of a graph is at most its pathwidth plus one [DMW02]. Many interesting graph families have bounded chromatic number and pathwidth at most $\mathcal{O}(\sqrt{n})$. Thus by Theorem 14.4 (b) they have $\mathcal{O}(n^{\frac{3}{2}})$ volume 3D grid drawings [DW04c]. Included in this family are planar graphs, graphs of bounded genus, graphs with no K_h-minor where h is a constant, and in fact all proper minor-closed families. Refer to Table 14.1 for details.

A vertex coloring is said to be a *strong star coloring*[DW04c] if, for each pair of color classes, all edges (if any) between them are incident to a single vertex. That is, each bichromatic subgraph consists of a star and possibly some isolated vertices. The *strong star chromatic number* of a graph G, denoted by $\chi_{\text{sst}}(G)$, is the minimum possible number of colors in a strong star coloring of G. No matter what ordering on the vertices in each color class in a strong star coloring, there is no X-crossing. Thus the track-number $\mathsf{tn}(G) \leq \chi_{\text{sst}}(G)$, as observed in [DW04c].

Every graph with m edges and maximum degree Δ has track number at most $14\sqrt{\Delta m}$. The proof relies on the Lovàsz Local Lemma [DW04c]. It is well-known that the chromatic number χ of a graph G is at most its maximum degree plus one. Together with Theorem 14.4 (b), this implies that graphs of bounded degree have 3D grid drawings with $\mathcal{O}(n^{\frac{3}{2}})$ volume.

Recently these results have been improved by essentially replacing Δ by the weaker notion of degeneracy. A graph G is *d-degenerate* if every subgraph of G has a vertex of degree at most d. The *degeneracy* of G is the minimum integer d such that G is d-degenerate. A d-degenerate graph is $(d+1)$-colorable by a greedy algorithm. For example, every forest is 1-degenerate, every outerplanar graph is 2-degenerate, and every planar graph is 5-degenerate. Dujmović and Wood proved that every m-edge d-degenerate graph G satisfies $(\mathsf{tn}(G) \leq)$ $\chi_{\text{sst}}(G) \leq 5\sqrt{2dm}$ and $(\mathsf{tn}(G) \leq)$ $\chi_{\text{sst}}(G) \leq (4 + 2\sqrt{2})m^{2/3}$. Again, Theorem 14.4 (b) implies that graphs of bounded degeneracy have 3D grid drawings with $\mathcal{O}(n^{\frac{3}{2}})$ volume.

The family of graphs with bounded degeneracy is vast. It includes all proper minor-closed families, such as, for example, planar graphs. In fact the family is strictly larger than that, since there are graph classes with bounded degeneracy but with unbounded clique minors. For example, the graph K'_n obtained from K_n by subdividing every edge once has degeneracy two, yet contains a K_n minor.

An affirmative answer to the following open problem would imply linear volume 3D grid drawings for planar graphs and thus an affirmative answer to Open Problem 14.1.

Open Problem 14.2 [DMW05] *Do planar graphs have $\mathcal{O}(1)$ track-number?*

A tight relationship between track layout and another well-studied type of graph drawing called queue layout has been established in [DPW04]. Queue layouts were introduced by Heath *et al.* [HLR92, HR92] and are defined as follows. A *queue layout* of a graph $G = (V, E)$ consists of a total order $<$ on the vertices $V(G)$, and a partition of the edges $E(G)$ into *queues*, such that no two edges in the same queue are *nested* with respect to $<$: two edges vw and xy are nested with respect to $<$ if $v < x < y < w$. The minimum number of queues in a queue layout of G is called the *queue-number* of G, and is denoted by $\mathsf{qn}(G)$.

It has been established in [DPW04] that a graph has a bounded track number if and only if it has a bounded queue number. Thus Open Problem 14.2 is equivalent to following open problem from 1992 due to Heath *et al.* [HLR92, HR92].

Open Problem 14.3 [HLR92, HR92] *Do planar graphs have $\mathcal{O}(1)$ queue-number?*

The best known upper bound for the queue-number of planar graph is $\mathcal{O}(log^4 n)$, due to Di Battista *et al.* [BFP10]. Unfortunately, for more general proper minor closed families, the best known bound for both the track number and the queue number is $\mathcal{O}(\sqrt{n})$. The bound follows easily from the fact that proper minor closed families have pathwidth bounded by $\mathcal{O}(\sqrt{n})$.

The best known bounds on the volume of 3D grid drawings for different graph families are summarized in Table 14.1.

Although almost all of the results on 3D grid drawings focus on the volume of such drawings, some results about aspect ratio of 3D grid drawings were reported in [DMW02].

3D grid drawings have been generalized in a number of ways.

Crossings allowed: Pór and Wood [PW04] considered a variation of 3D grid drawings where edges are allowed to cross. Specifically, they considered 3D drawings where each vertex is represented by a distinct grid point in $\mathbb{Z}^3$ such that the line-segment representing each edge does not intersect any vertex, except the two at the endpoints of the edge. With that relaxation, better volume bounds are possible. For instance, a 3D drawing of the complete graph K_n is nothing more than a set of n gridpoints with no three collinear, and such a set can be found with grid volume $\Theta(n^{\frac{3}{2}})$ [PW04]. Generalizing this construction, Pór and Wood [PW04] proved that if edge crossings are allowed, every c-colorable graph has a 3D drawing with $\mathcal{O}(n\sqrt{c})$ volume. That bound is optimal for the c-partite Turán graph.

14.2.2 Upward

Another straight-line graph drawing model for the 3D integer grid is the upward 3D grid drawing. A 3D grid drawing of a directed graph G is *upward* if $\mathsf{z}(v) < \mathsf{z}(w)$ for every arc vw of G. Obviously an upward 3D grid drawing can only exist if G is a directed acyclic graph (a *dag*). Upward two dimensional drawings have been widely studied.

Poranen [Por00] proved that series-parallel digraphs have upward 3D grid drawings with $\mathcal{O}(n^3)$ volume, and that this bound can be improved to $\mathcal{O}(n^2)$ and $\mathcal{O}(n)$ in certain special cases.

Di Giacomo *et al.* [dGLMW05] extended the definition of track layouts to dags as follows. An *upward track layout* of a dag G is a track layout of the underlying undirected graph of G, such that if G^+ is the directed graph obtained from G by adding an arc from each vertex v to the successor vertex in the track that contains v (if it exists), then G^+ is still acyclic. The upward track number of G, denoted by $\mathsf{utn}(G)$, is the minimum integer t such that G has an upward t-track layout. Di Giacomo *et al.* [dGLMW05] proved the following analogue of Theorem 14.4 (a).

Theorem 14.5 [dGLMW05] *Let G be an n-vertex graph with upward track-number $\mathsf{utn}(G) \leq t$. Then G has an $\mathcal{O}(t) \times \mathcal{O}(t) \times \mathcal{O}(tn)$ upward 3D grid drawing with $\mathcal{O}(t^3 n)$ volume. Conversely, if a dag G has an $X \times Y \times Z$ upward 3D drawing then G has upward track-number $\mathsf{utn}(G) \leq 2XY$.*

This theorem provides motivation for studying upward track layouts of dags. Di Giacomo *et al.* [dGLMW05] proved that directed trees have upward track number at least four and at most seven. The upper bound was subsequently improved to five [DW06]. Together with the above theorem, that implies that all directed trees have upward 3D grid drawings with linear volume [dGLMW05]. Although undirected outerplanar graphs (and all bounded treewidth graphs) have bounded track number and linear volume 3D grid drawings, the situation is much different in the case of dags. In particular, Di Giacomo *et al.* [dGLMW05] proved that there is an outerplanar dag that requires $\Omega(n^{3/2})$ volume in every upward 3D grid drawing. In particular, as illustrated in Figure 14.4, let G_n be the dag with vertex set $\{u_i : 1 \leq i \leq 2n\}$ and arc set $\{\overrightarrow{u_i u_{i+1}} : 1 \leq i \leq 2n-1\} \cup \{\overrightarrow{u_i u_{2n-i+1}} : 1 \leq i \leq n\}$.

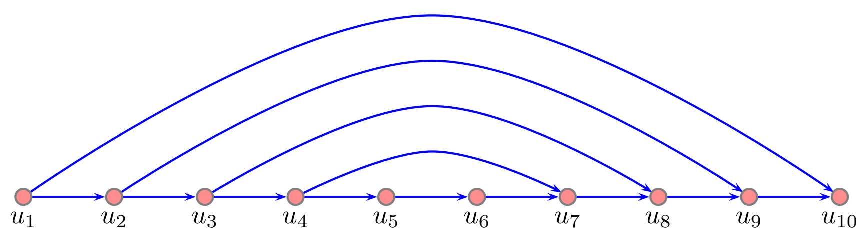

Figure 14.4 Illustration of G_5.

Suppose that G_n has an $X \times Y \times Z$ upward 3D grid drawing. Observe that G_n is outerplanar and has a Hamiltonian directed path $(u_1, u_2, \ldots, u_{2n})$. Thus $(u_1, u_2, \ldots, u_{2n})$ is the only topological ordering of G_n. Thus $Z \geq 2n$. Di Giacomo *et al.* [dGLMW05] proved that $\mathsf{utn}(G_n) \geq \sqrt{2n}$. Theorem 14.5 implies that $2XY \geq \mathsf{utn}(G_n) \geq \sqrt{2n}$. Hence the volume is $\Omega(n^{3/2})$ [dGLMW05].

This result highlights a substantial difference between 3D grid drawings of undirected graphs and upward 3D grid drawings of dags, since every (undirected) outerplanar graph has a 3D grid drawing with linear volume [FLW01]. In the full version of their paper, Di Giacomo *et al.* [dGLMW05] constructed an upward 3D grid drawing of G_n with $\mathcal{O}(n^{3/2})$ volume. It is unknown whether every n-vertex outerplanar dag has an upward 3D grid drawing with $\mathcal{O}(n^{3/2})$ volume.

The proof that every graph has a 3D grid drawing with $\mathcal{O}(n^3)$ volume [CELR96] generalizes to upward 3D grid drawings. In particular,

Theorem 14.6 [DW06] *Every dag G on n vertices has a $2n \times 2n \times n$ upward 3D grid drawing with $4n^3$ volume. Moreover, the bounding box of every upward 3D grid drawing of the complete dag on n vertices is at least $\frac{n}{4} \times \frac{n}{4} \times n$, and thus has $\Omega(n^3)$ volume.*

As already stated, Pach *et al.* [PTT99] proved that every c-colorable graph has an $\mathcal{O}(c) \times \mathcal{O}(n) \times \mathcal{O}(cn)$ drawing with $\mathcal{O}(c^2n^2)$ volume. The result generalizes to upward 3D grid drawings as follows.

Theorem 14.7 [DW06] *Every n-vertex c-colorable dag G has a $c \times 4c^2n \times 4cn$ upward 3D grid drawing with volume $\mathcal{O}(c^4n^2)$.*

Every acyclic orientation of $K_{n,n}$ requires $\mathcal{O}(n^2)$ volume in every upward 3D grid drawing [PTT99]. Hence Theorem 14.7 is tight for constant c. The theorem implies the quadratic volume upper bound for numerous families of dags, including series-parallel dags, planar dags, dags of constant treewidth, all proper minor-closed dags, dags with bounded degeneracy, and so on.

14.2.3 Polyline

Consider a relaxation of 3D straight-line grid drawings where edges are allowed to have bends. In particular, a *three-dimensional polyline grid drawing* of a graph, henceforth called a *3D polyline drawing*, represents the vertices by distinct gridpoints, and represents each edge as a polygonal chain between its endpoints with bends (if any) also at gridpoints, such that distinct edges only intersect at common endpoints, and each edge only intersects a vertex that is an endpoint of that edge. Here a point where a polygonal chain changes its direction is called a *bend*. A 3D polyline drawing with at most b bends per edge is called a *3D b-bend drawing*. Thus 0-bend drawings are 3D grid drawings.

As discussed in the next section, the volume and number of bends in 3D polyline drawings where edges are restricted to be axis-aligned have been studied extensively. The study of 3D polyline drawings has only recently been initiated [DW04b]. Tools developed for 3D (straight-line) grid drawings, such as track layouts, turned out to be useful for the polyline drawings as well. That is simply because a 3D b-bend drawing of a graph G is precisely a 3D straight-line drawing of a subdivision of G with at most b division vertices per edge. This provides a motivation for a study of track layouts of graph subdivisions. Recall that a *subdivision* of a graph G is a graph D obtained from G by replacing each edge $vw \in E(G)$ by a path having v and w as endpoints and having at least one edge. Internal vertices on this path are called *division* vertices.

Dujmović and Wood [DW04b] proved that every n-vertex m-edge graph G has a subdivision D with at most $\log n$ division vertices per edge and such that the track number of D is at most four. Thus by the aforementioned relationship to the 3D grid drawings, D has a (straight-line) 3D grid drawing with $\mathcal{O}(|V(D)|)$ volume. Since $|V(D)| = m \log n$, it follows that every graph G has a 3D polyline drawing with $\mathcal{O}(m \log n)$ volume and at most $\log n$ bends per edge. These results are further generalized [DW04b] as indicated in Table 14.1. For example, complete graphs admit 2-bend 3D polyline grid drawings in $\mathcal{O}(n^2)$ volume. That bound is best possible if the number of bends per edge is restricted to be at most two. If only one bend per edge is allowed, then the complete graphs admit 1-bend 3D polyline grid drawings with $\mathcal{O}(n^{5/2})$ [DEL+05] volume. The best known lower bound in this case is $\Omega(n^2)$.

Table 14.1 summarizes the best known upper bounds on the volume and bends per edge in 3D grid drawings and 3D polyline drawings. In general, there is a trade-off between few bends and small volume in such drawings, which is evident in Table 14.1.

graph family	bends per edge	volume	reference
straight-line			
arbitrary	0	$\mathcal{O}(n^3)$	[CELR96]
arbitrary	0	$\mathcal{O}(m^{4/3}n)$	[DW04c]
maximum degree Δ	0	$\mathcal{O}(\Delta mn)$	[DW04c]
maximum degree Δ	0	$\mathcal{O}(\Delta^{15/2}m^{1/2}n)$	[DW06]
d-degenerate	0	$\mathcal{O}(dmn)$	[DW06]
d-degenerate	0	$\mathcal{O}(d^{15/2}m^{1/2}n)$	[DW04c]
c-colorable	0	$\mathcal{O}(c^2n^2)$	[PTT99]
c-colorable	0	$\mathcal{O}(c^6m^{2/3}n)$	[DW04c]
proper minor-closed	0	$\mathcal{O}(n^{3/2})$	[DW04c]
planar	0	$\mathcal{O}(n\log^{16} n)$	[BFP10]
outerplanar	0	$\mathcal{O}(n)$	[FLW01]
bounded treewidth	0	$\mathcal{O}(n)$	[DMW05]
polyline			
c-colorable q-queue	1	$\mathcal{O}(cqm)$	[DW04b]
arbitrary	1	$\mathcal{O}(nm)$	[DW04b]
arbitrary	1	$\mathcal{O}(n^{5/2})$	[DEL+05]
q-queue	2	$\mathcal{O}(qn)$	[DW04b]
q-queue (constant $\epsilon > 0$)	$\mathcal{O}(1)$	$\mathcal{O}(mq^{\epsilon})$	[DW04b]
q-queue	$\mathcal{O}(\log q)$	$\mathcal{O}(m\log q)$	[DW04b]

Table 14.1 Volume of 3D straight-line and polyline drawings of graphs with n vertices and $m \geq n$ edges.

In the case of dags, upward variants of 3D polyline grid drawings have also been considered. For instance, with two bends per edge allowed, every n-vertex dag G has an upward 2-bend $n \times 2 \times 2n$ 3D grid drawing with volume $4n^2$ [DW06].

14.3 Orthogonal grid drawings

3D polyline (b-bend) drawings where all edge segments are restricted to be parallel to one of the three axes are called *3D orthogonal* (b*-bend*) *point-drawings*. This restriction implies that only graphs with maximum degree at most six have such drawings. For that reason the notion is generalized to *3D orthogonal* (b*-bend*) (*box-*)*drawings*, where vertices of the graph are represented by pairwise non-intersecting boxes. A *box* is a rectanguloid with all of its corners at grid points. A 3D orthogonal (b-bend) (box)-drawing where all boxes degenerate to cubes, line-segments, or points is called, respectively, a *3D orthogonal* (b*-bend*) *cube-, line-*, or *point-drawing*.

The 3D orthogonal drawings have very good angular resolution, which makes them suitable for numerous applications. Minimum edge separation and minimum vertex separation are also guaranteed in such drawings. Notice that neither good angular resolution nor good edge separation is a feature of 3D (straight-line) grid drawings. The main quality measures for 3D orthogonal drawings are the volume and the number of bends (per edge). Other criteria of importance include the length of the edges, and, in the case of 3D orthogonal box-drawings, the size and the shape of the boxes. While the focus of this section are orthogonal drawings in 3D, degree-4 graphs admit 3D polyline drawings with angular resolution even better than 90 degrees. Study of such drawings with small number of bends and good volume bounds has recently been initiated by Eppstein *et al.* [ELMN11].

It is $\mathcal{NP}$-hard to optimize most of these aesthetic criteria for 3D orthogonal drawings. Using straightforward extensions of known two-dimensional hardness results, Eades *et al.* [ESW96] showed that it is $\mathcal{NP}$-hard to find a 3D orthogonal point-drawing of a graph that minimizes any one of the following aesthetic criteria: the volume, the number of bends per edge, the total number of bends, and the total edge length.

Not surprisingly, the 3D orthogonal point-drawings were the first to be studied; we consider them in the next section, followed by a review of 3D orthogonal box-drawings in Section 14.3.2.

14.3.1 Point-drawings

In a 3D orthogonal point-drawing a vertex can have at most six neighbours. Thus only graphs of degree at most six may admit such drawings. In fact a graph has a 3D orthogonal point-drawing if and only if its maximum degree is at most six. This result will be discussed shortly (Theorem 14.8 below). The drawings used in establishing this result have many bends. This is unavoidable, since every 3D orthogonal point-drawing of the triangle (that is, K_3) obviously has at least one bend. Moreover, to draw an edge between any pair of vertices not on the same grid line, at least one bend is required, and to draw and edge between a pair not on the same grid plane, at least two bends are required. This sheds light on the fact that no nontrivial class of graphs (excluding trees) is known to admit 3D orthogonal point-drawings with zero bends. Less obvious is the well-known result that any 3D orthogonal point-drawing of a multi-graph comprised of two vertices and six edges has an edge with at least three bends. For simple graphs, K_5 requires an edge with at least two bends [Woo03a]. This provides the best known lower bound on the number of bends per edge for 3D orthogonal point-drawings of degree-6 graphs.

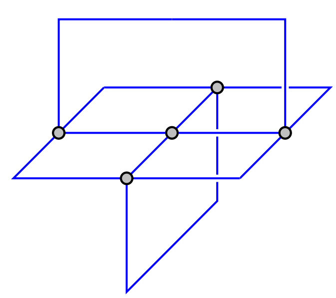

Figure 14.5 3D orthogonal 2-bend point-drawing of K_5 (in coplanar model).

Volume $\Theta(n^{3/2})$:

One of the earliest results concerning 3D orthogonal point-drawings is due to Kolmogorov and Barzdin [KB67] and established a lower bound of $\Omega(n^{3/2})$ for the volume of degree-6 graphs. This lower bound was matched with an upper bound by Eades *et al.* [ESW96] to establish the following theorem.

Theorem 14.8 [ESW96, KB67] *Every n-vertex degree-6 graph has a 3D orthogonal point-drawing in $\mathcal{O}(n^{3/2})$ volume, and that bound is best possible for some degree-6 graphs.*

To obtain the upper bound, Eades *et al.* [ESW96] developed an $\mathcal{O}(n)$-time algorithm[2] that produces a 3D orthogonal point-drawing for a degree-6 graph G. Their algorithm is a modification of the method developed by Kolmogorov and Barzdin [KB67] for a similar problem. The algorithm places all the vertices of G on an $\mathcal{O}(n) \times \mathcal{O}(n)$ grid in the $\mathbb{Z} = 0$ plane and draws each edge with at most sixteen bends. This model of drawing where all the vertices intersect one grid plane is know as the *coplanar* model. Figure 14.5 illustrates a 2-bend orthogonal point-drawing of K_5 in the coplanar model.

2 and 3 Bends:

Theorem 14.8 states that for the point-drawings, the optimal volume for degree-6 graphs is known (at least asymptotically). The situation is different for the number of bends per edge. As noted above two bends per edge may be necessary. The best known upper bound is three. This result was first proved by Eades *et al.* [ESW00].

Theorem 14.9 [ESW00] *Every degree-6 graph has a 3D 3-bend orthogonal point-drawing.*

We now overview the most commonly used approach for producing 3D orthogonal point-drawings. The approach was first taken by Eades *et al.* [ESW00] in their 3-bend algorithm that establishes Theorem 14.9.

A *cycle cover* of a graph G, also called a 2-*factor*, is a 2-regular spanning subgraph of G, that is, a spanning subgraph that consists of cycles. If the graph is directed, then the cycles in the cover are required to be directed as well. Eades *et al.* [ESW00] gave an algorithmic proof that the edges of every degree-6 graph G can be oriented in such a way that G is a subgraph of some directed graph G' (possibly with loops) such that the edges of G' can be colored with three colors each of which induces a directed cycle cover of G'. The proof can be viewed as a repeated application of the classical result of Petersen that every regular graph of even degree has a 2-factor. The cycle covers can be computed in $\mathcal{O}(n)$ time for n-vertex graphs.

Having this in mind, most algorithms for producing 3D orthogonal point-drawings start off with the decomposition of G' into three cycle covers, denoted, say, by $\mathcal{C}_{red}$, $\mathcal{C}_{blue}$, and $\mathcal{C}_{green}$. In the second step vertices of G' are positioned on the 3D grid in some way that makes drawing the red cycles easy. For example, in the coplanar model, vertices can be placed in the $\mathbb{Z} = 0$ plane and all red edges can be drawn in that plane. The remaining edges $\mathcal{C}_{blue}$ and $\mathcal{C}_{green}$ are then routed above and below the $\mathbb{Z} = 0$ plane, respectively. In general, the third step involves finding drawings for the edges in $\mathcal{C}_{blue}$ and $\mathcal{C}_{green}$.

The 3-bend algorithm of Eades *et al.* [ESW00] positions each vertex v_i of G' at $(3i, 3i, 3i)$ for some arbitrary vertex ordering $(v_1, v_2, \ldots v_n)$ of $V(G')$. This model of 3D orthogonal point-drawings, where vertices are place along the 3D diagonal of a cube, is called the *diagonal model*. The resulting drawings have volume at most $8n^3$ after all the grid planes not containing a vertex or a bend are deleted. Wood [Woo04] modifies the 3-bend algorithm of Eades *et al.* [ESW00] to produce 3-bend drawings in the diagonal model with $n^3 + o(n^3)$

[2]The running time in the conference paper is $\mathcal{O}(n^{3/2})$. This was later reduced in [ESW00].

volume, which is to date the best volume bound on 3D orthogonal 3-bend drawings. To achieve this, Wood places each vertex v_i of G' at (i, i, i) in a particular vertex ordering $(v_1, v_2, \ldots v_n)$ stemming from book embeddings. For more on book embeddings, refer to the next section on graph thickness. While the algorithm of Eades *et al.* runs in $\mathcal{O}(n)$ time, the algorithm of Wood runs in $\mathcal{O}(n^{5/2})$ time due to the book embedding computation. The diagonal model was also used in the incremental algorithm of Papakostas and Tollis [PT99]. Their algorithm, which runs in $\mathcal{O}(n)$ time, supports on-line insertion of vertices in constant time. The resulting 3D orthogonal 3-bends point-drawings have volume at most $4.63n^3$.

The upper bound from Theorem 14.9 and the lower bound of two on the number of bends per edge leave the following open problem.

Open Problem 14.4 [ESW00] *Does every degree-*6 *graph have a 3D 2-bend orthogonal point-drawing?*

This problem is considered to be the most important open problem concerning 3D orthogonal point-drawings. The answer to the question remains unknown even when attention is restricted to more specific classes of graphs, including degree-6 planar graphs, degree-6 series-parallel graphs, and degree-6 outerplanar graphs. It is easy to observe that every degree-6 tree has a 3D orthogonal point-drawing with no bends.

A natural candidate for answering Open Problem 14.4 in the negative was K_7, as conjectured in the conference version of [ESW00]. The counterexample to that conjecture was discovered by Wood [Woo03a]. His construction is illustrated in Figures 14.6 and 14.7 (courtesy of David R. Wood). Moreover, Wood exhibited 3D 2-bend point-drawings for other small multipartite 6-regular graphs: $K_{6,6}$, $K_{3,3,3}$ and $K_{2,2,2,2}$.

For degree-5 graphs, Wood [Woo03b] answered Open Problem 14.4 in the affirmative.

Theorem 14.10 [Woo03b] *Every degree-*5 *graph has a 3D 2-bend orthogonal point-drawing.*

The $\mathcal{O}(n^2)$-time algorithm of Wood that establishes this result produces 3D orthogonal point-drawings of degree-6 graphs in the so called *general position model*, where no pair of vertices belongs to the same grid plane. (Note, for example, that a drawing in the diagonal model is also in the general position model.) In the case of degree-5 graphs, the algorithm outputs 2-bend drawings in the general position model. While this model allows for 2-bend drawings for degree-5 graphs, the same is not the case for degree-6 graphs. In particular, Wood [Woo03a] constructed an infinite family of degree-6 graphs that have an edge with at least 3 bends in every 3D orthogonal point-drawing in the general position model.

Tradeoffs and more bounds:

Tradeoff issues between the maximum number of bends per edge and the volume of 3D orthogonal point-drawings were first studied by Eades *et al.* [ESW00]. They began with an algorithm to draw a degree-6 graph in the coplanar model with $\mathcal{O}(n^{3/2})$ volume and at most 7 bends per edge. By successive refinements of this algorithm, they obtained 3D orthogonal point-drawings of degree-6 graphs with the following bounds: volume $\mathcal{O}(n^2)$ with at most 6 bends per edge, and volume $\mathcal{O}(n^{5/2})$ with at most 5 bends per edge. For drawings in $\mathcal{O}(n^2)$ volume, Biedl [BJSW01] reduced the number of bends per edge to 4.

Numerous refinements of these results have appeared in the literature. Table 14.2 summarizes the best known bounds on 3D orthogonal point-drawings. Some of the algorithms associated with the bounds in Table 14.2 are dynamic, supporting operations such as vertex insertion [PT99, CGJW01] and deletion, as well as edge deletion and insertion [CGJW01].

In addition to the number of bends per edge, the total number of bends in 3D orthogonal point-drawings has also been investigated. Wood [Woo03a] showed that every 3D orthogonal

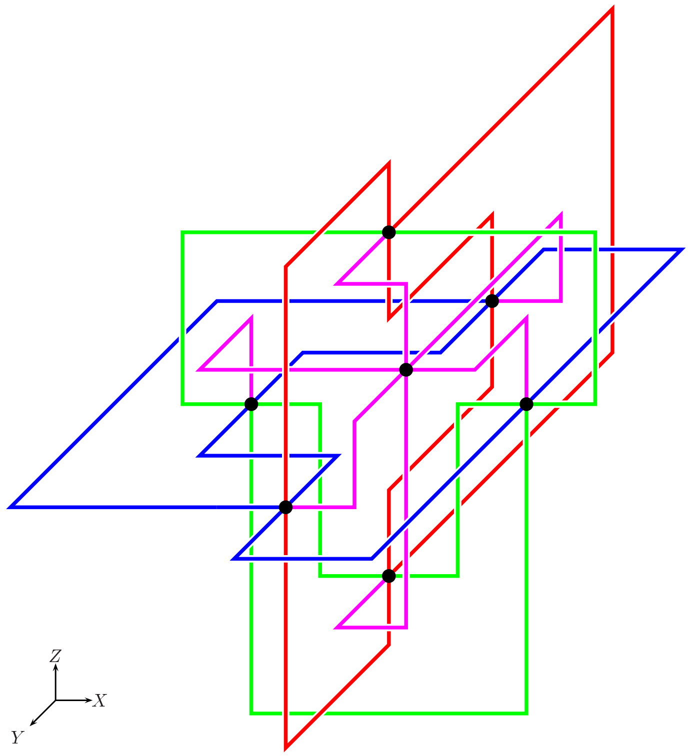

Figure 14.6 A 3D orthogonal 2-bend point-drawing of K_7 [Woo03a].

point-drawing of K_7 has at least 20 bends, which implies the lower bounds of $20m/21$ bends for simple m-edge graphs. The algorithm of Wood [Woo03b] that establishes Theorem 14.10 also produces 3D orthogonal point-drawings for simple m-edge degree-6 graphs with at most $16m/7$ bends, thus having an average of $2\frac{2}{7}$ bends per edge. The drawings are in the general position model, for which the bound is optimal since K_7 requires $\frac{16}{7}|E(K_7)|$ bends in that model, as established in [Woo03a].

graph family	max. (avg.) bends per edge	volume	reference
multigraph	7	$\Theta(n^{3/2})$	[ESW00]
multigraph (dynamic)	14	$\Theta(n^{3/2})$	[BJSW01]
multigraph	4	$\mathcal{O}(n^2)$	[BJSW01]
multigraph (dynamic)	5	$\mathcal{O}(n^2)$	[CGJW01]
multigraph $\Delta \leq 4$	3	$\mathcal{O}(n^2)$	[ESW00]
simple	4 $(2\frac{2}{7})$	$2.13n^3$	[Woo03b]
multigraph (dynamic)	3	$4.63n^3$	[PT99]
multigraph	3	$n^3 + o(n^3)$	[Woo04]
simple $\Delta \leq 5$	2	n^3	[Woo03b]

Table 14.2 The volume and the number of bends per edge in 3D orthogonal point-drawings of n-vertex graphs with maximum degree $\Delta \leq 6$.

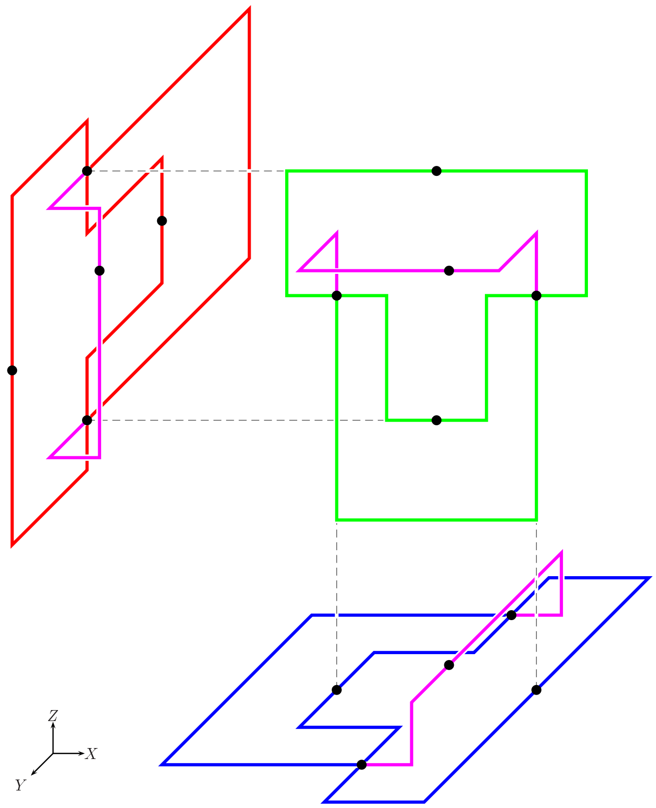

Figure 14.7 Breakaway view of the 3D orthogonal 2-bend point-drawing of K_7 [Woo03a].

14.3.2 Box-drawings

Only degree-6 graphs admit 3D orthogonal point-drawings. Hence it was only natural to consider the extension to box-drawings for general graphs. For point-drawings, it was enough to consider K_3 to realize that there are degree-6 graphs that do not admit such drawings with straight-line edges. It is less obvious that not all graphs admit 3D orthogonal box-drawings with straight-line edges (that is, with zero bends). In a straight-line orthogonal box-drawing of a graph G, each edge is a line segment parallel to one of the three axes. This defines an associated coloring of the edges with three colors, where a subgraph of G induced by each color class has a visibility representation by rectangles. (Refer to the last section, page 24, for the definition of a visibility representation.) Bose *et al.* [BEF$^+$98] proved that K_n does not have such a representation for $n \geq 56$. Ramsey theory implies that for every constant $c \in \mathbb{N}$ there is a constant $r(c)$ (the Ramsey number) such that every edge 3-coloring of the complete graph K_n with $n \geq r(c)$ contains a monochromatic subgraph isomorphic to K_c. With $c = 56$, that establishes the fact that $K_{r(56)}$ does not have a straight-line 3D orthogonal box-drawing. This argument (in three and higher dimensions) was first pointed out by Biedl *et al.* [BSWW99]. The constant $r(56)$, stemming from Ramsey theory, is a truly big number. Fekete and Meijer [FM99] significantly improved that upper bound to K_{184}. Their proof uses the fact that K_{56} does not have a 3D rectangle visibility representation. The largest complete graph known to admit a straight-line 3D orthogonal box-drawing is K_{56} [FM99].

The above discussion highlights that not all graphs have 3D orthogonal box-drawings with zero bends. Indeed, it is easy to observe that every n-vertex m-edge graph G has an orthogonal (line)-drawing with one bend per edge: simply represent each vertex v_i, $1 \leq i \leq n$, of G by a line-segment with endpoints $(i, i, 1)$ and (i, i, m), and then draw each edge in distinct $\mathsf{Z} = j$ planes, $1 \leq j \leq m$, using one bend. The resulting drawing has $\mathcal{O}(n^2 m)$ volume. Better volume bounds are possible for 3D orthogonal 1-bend box-drawings. Biedl *et al.* [BSWW99] showed that in the previous construction with the segments having endpoints at $(i, i, 1)$ and (i, i, n), it is possible to draw all the edges of K_n in $\mathsf{Z} = j$, $1 \leq j \leq m$, using one bend per edge. They suggested a relationship between assigning edges to the planes in this type of drawing and assigning edges to the pages of a book embedding. This relationship was later explored by Wood [Woo01], resulting in improved volume bounds for 1-bend box-drawings of m-edge graphs. In particular, he proved that every graph has a 3D orthogonal 1-bend box-drawing in $\mathcal{O}(n^{3/2} m)$ volume.

A lower bound of $\Omega(n^{5/2})$ for the volume of 3D orthogonal box-drawings of n-vertex graphs (regardless of the number of bends) was established by Biedl *et al.* [BSWW99]. They developed an $\mathcal{O}(m)$-time algorithm that constructs drawings matching that volume bound and using at most 3 bends per edge, thus establishing that all n-vertex graphs have 3D orthogonal 3-bend box-drawings in $\Theta(n^{5/2})$ volume. Closing the gap between the $\mathcal{O}(n^3)$ upper bound and the $\Omega(n^{5/2})$ lower bound for 3D orthogonal 1-bend box-drawings of K_n remains an interesting open problem.

The lower bound of Biedl *et al.* [BSWW99] was established using the complete graph K_n. The proof relies critically on the fact that between any two disjoint vertex sets of size $\Omega(n)$ in K_n, there are $\Theta(n^2)$ edges. To generalize this lower bound to sparse graphs and to be able to express it in terms of the number of edges, Biedl *et al.* [BTW06] exhibited graphs such that between any two disjoint vertex sets of size $\Omega(n)$ there are $\Theta(m)$ edges. That allowed them to extend the arguments of [BSWW99] to establish the lower bound of $\Omega(m\sqrt{n})$ on the volume of 3D orthogonal box-drawings of m-edge n-vertex graphs. They developed an $\mathcal{O}(m^2/\sqrt{n})$-time algorithm that constructs drawings matching that volume bound and using at most 4 bends per edge, thus establishing that all graphs have 3D orthogonal 4-bend box-drawings in $\Theta(m\sqrt{n})$ volume. It is unknown whether all m-edge graphs admit 3D orthogonal box-drawings with such volume and at most 3 bends per edge, as is the case for K_n.

The discussion above pertains to drawings where the volume and the number of bends per edge are the only concerns. The shapes and the sizes of boxes used to represent vertices are unrestricted. However, for box-drawings the size and the shape of a vertex with respect to its degree are also important aesthetic criteria. For a vertex v in a 3D orthogonal box-drawing the *surface* of v is the number of grid lines intersecting the box-representing v times two. The surface of v indicates the number of grid lines available for drawing edges incident to v. In point-drawings, for example, the surface of each vertex is six. Generally, in any 3D orthogonal box-drawing, the surface of each vertex v is at least the degree of v. Ideally, the surface of v should also not be much bigger than the degree of v. Biedl *et al.* [BTW06] defined a 3D orthogonal box-drawing of a graph G to be *degree-restricted* if there exists some constant $\alpha \geq 1$ such that for every vertex v in G, surface$(v) \leq \alpha \cdot$degree(v).

Degree restricted drawings do not, however, impose any aesthetic restriction on the shape of the boxes used to represent vertices. The *aspect ratio* of a vertex in a 3D orthogonal box-drawing is the ratio of the length (measured in the number of grid points) of the longest side of the box representing that vertex to the shortest side of that box. 3D orthogonal box-drawings have a *bounded vertex-aspect ratio* if there exists a constant r such that all vertices have aspect ratios at most r. Note that $r \geq 1$, and for the case of 3D orthogonal point-drawings and cube-drawings, it is one. Also note that degree-restricted drawings may have unbounded vertex-aspect ratio; consider, for example, a drawing in which each vertex is represented by a segment with length equal to its degree.

The discussion at the beginning of this subsection pertains to 3D orthogonal box-drawings with (possibly) unbounded vertex-aspect ratios and with no degree-restrictions. The best known upper bounds on the volume and the number of bends per edge in such unrestricted 3D orthogonal box-drawings are summarized in the top part of Table 14.3. The upper bounds can be compared to the best known lower bound on the volume of such drawings which, as discussed above, is $\Omega(m\sqrt{n})$ regardless of the number of bends [BTW06]. The table exhibits the tradeoff between the number of bends per edge and the volume of such drawings.

Biedl *et al.* [BTW06] derived lower bounds for the volume of 3D orthogonal box-drawings that are required to be degree-restricted and/or have bounded vertex-aspect ratio. In particular, they proved an $\Omega(m^{3/2}/\alpha)$ lower bound on the volume of 3D orthogonal box-drawings that are degree-restricted for some $\alpha \geq 1$, as well as an $\Omega(m^{3/2}/\sqrt{r})$ lower bound on the volume of 3D orthogonal box-drawings for which each vertex has aspect ratio at most r. For bounded α and bounded r, both bounds become $\Omega(m^{3/2})$. The discussion pertaining to the proof technique of Biedl *et al.* [BTW06] used to derive the $\Omega(m\sqrt{n})$ volume bound for unrestricted drawings applies to these two lower bounds as well.

Biedl *et al.* [BTW06] also developed an algorithm that constructs the corresponding 3D orthogonal box-drawings matching the volume lower-bound and using at most 6 bends per edge, thus establishing that all m-edge graphs have 3D orthogonal 6-bend box-drawings with volume $\Theta(m^{3/2})$ such that the drawings are degree-restricted and have bounded aspect ratio.

The best known upper bounds on the volume and the number of bends per edge in degree-restricted 3D orthogonal box-drawings are summarized in the middle part of Table 14.3, while drawings that are both degree-restricted and have bounded vertex-aspect ratio are addressed at the bottom of the table. These upper bounds on the volume can be compared to the best known lower bound of $\Omega(m^{3/2})$.

The table reveals that no further asymptotic improvements are possible for the volume of drawings in all three aesthetic models discussed. There is room for improvement, however, with regard to the number of bends per edge, as suggested by some of the open problems mentioned in this subsection.

14.4 Thickness

Thickness is a classical graph parameter that has been studied since the early 1960s. It was first defined by Tutte [Tut63]. The *thickness* of a graph G, denoted by $\theta(G)$, is the minimum $k \in \mathbb{N}$ such that the edge set of G can be partitioned into k planar subgraphs.

For ease of exposition in this section, we express the concept of thickness in terms of drawings in the plane. The *thickness* of a drawing in the plane with vertices represented as points and edges represented as simple curves is the minimum $k \in \mathbb{N}$ such that the edges of the drawing can be partitioned into k subgraphs such that each subgraph has no crossings in the drawing; that is, each edge is assigned one of k colors such that no pair of like-colored edges of the drawing cross. Since any planar graph can be drawn with its vertices at prespecified points in the plane (see, for example, [PW01]), a graph has thickness k if and only if it has a drawing in the plane with thickness k [Hal91]. However, in such a drawing the edges may be highly curved and thus unsuitable for most applications. For instance, when the edges are represented by polygonal chains, then $\Omega(n)$ bends per edge may be needed [PW01]. This motivates the notion of geometric thickness.

A drawing of a graph in the plane is *geometric* if every edge is represented by a straight

graphs	bends	volume	reference
unbounded vertex-aspect ratio / not degree-restricted			
simple	1	$\mathcal{O}(n^3)$	[BSWW99]
simple	1	$\mathcal{O}(n^{3/2}m)$	[Woo01]
simple	2	$\mathcal{O}(nm)$	[Woo01]
simple	3	$\mathcal{O}(n^{5/2})$	[BSWW99]
multigraphs	3	$\mathcal{O}(nm)$	[BTW06]
simple	4	$\Theta(m\sqrt{n})$	[BTW06]
unbounded vertex-aspect ratio / degree-restricted			
simple	2	$\mathcal{O}(n^2m)$	[Bie98, Woo99]
simple	2	$\mathcal{O}(n^2\Delta)$	[Bie98]
multigraphs	5	$\mathcal{O}(m^2)$	[BTW06]
multigraphs	6	$\Theta(m^{3/2})$	[BTW06]
bounded vertex-aspect ratio / degree-restricted			
simple	2	$\mathcal{O}((nm)^{3/2})$	[Bie98, Woo99]
simple	2	$\mathcal{O}(nm\sqrt{\Delta})$	[Bie98]
multigraphs	5	$\mathcal{O}(m^2)$	[BTW06]
simple	10	$\mathcal{O}((n\Delta)^{3/2})$	[HTS83]
multigraphs	6	$\Theta(m^{3/2})$	[BTW06]

Table 14.3 Volume and the maximum number of bends in 3D orthogonal (box)-drawings of n-vertex m-edge degree-Δ graphs for various aesthetic criteria.

line-segment. The *geometric thickness* of a graph G, denoted by $\overline{\theta}(G)$, is the minimum $k \in \mathbb{N}$ such that there is a geometric drawing of G with thickness k. Kainen [Kai73] first defined geometric thickness under the name of *real linear thickness*, and it has also been called *rectilinear thickness*. By the Fáry-Wagner theorem, a graph has geometric thickness one if and only if it is planar. Graphs of geometric thickness two, the so-called *doubly linear* graphs, were studied by Hutchinson *et al.* [HSV99] in the context of rectangle-visibility graphs.

Another parameter closely related to geometric thickness is book thickness. A geometric drawing in which the vertices are in convex position is called a *book embedding*. The *book thickness* of a graph G, denoted by $\mathsf{bt}(G)$, is the minimum $k \in \mathbb{N}$ such that there is book embedding of G with thickness k. The book embeddings have also been called *stack layouts*, and book thickness is also called *stacknumber*, *pagenumber* and *fixed outerthickness*.

Whether two edges cross in a book embedding is simply determined by the relative positions of their endpoints in the cyclic order of the vertices around the convex hull. One can think of the vertices as being ordered on the spine of a book and each plane subgraph being drawn without crossings on a single page. A graph has book thickness one if and only if it is outerplanar [BK79]. Bernhart and Kainen[BK79] proved that a graph has book thickness at most two if and only if it is a subgraph of a Hamiltonian planar graph. Unlike thickness, being able to partition the edge set of a graph G into k outerplanar subgraphs does not imply that G has book thickness at most k. For example, the edge set of K_5 can be partitioned into two cycles, yet K_5 has book thickness more than two, since it is not a subgraph of a Hamiltonian planar graph. The situation is similar for geometric thickness as will soon become clear.

Book embeddings, first defined by Ollmann [Oll73], are ubiquitous structures with a variety of applications; see [DW04a] for a survey with over 50 references. These applications include sorting permutations, fault tolerant VLSI design, and compact graph encodings as well as graph drawing. In general, drawings arising from the study of thickness have applications in graph visualization (where each plane subgraph is colored by a distinct color), and in multilayer VLSI (where each plane subgraph corresponds to a set of wires that can be routed without crossings in a single layer).

First we consider the relationship between the three thickness parameters. By definition, for every graph G

$$\theta(G) \leq \overline{\theta}(G) \leq \mathsf{bt}(G). \tag{14.1}$$

These inequalities have been shown to be strict for certain graphs [DEH00]. In the other direction, no such relationship is possible for any bounding function. Eppstein [Epp01] proved that geometric thickness is not bounded by any function of book thickness. In particular, the graph obtained by subdividing each edge of K_n once has geometric thickness at most two. On the other hand, a Ramsey-theoretic argument shows that the book thickness of that graph is not bounded by any constant.

Using a more elaborate Ramsey-theoretic argument applied to graphs formed by starting with n points and adding a new point adjacent to each triple of the n points, Eppstein [Epp04a] proved that geometric thickness is not bounded by any function of thickness. In particular, for every t there exists a graph with thickness three and geometric thickness at least t. This leaves an interesting open problem.

Open Problem 14.5 [Epp04a] *Do graphs with thickness two have bounded geometric thickness?*

Complete graphs: The thickness of the complete graph K_n was intensely studied in the 1960's and 1970's. Results by a number of authors [AG76, Bei67, BH65, May72] together prove that $\theta(K_n) = \lceil (n+2)/6 \rceil$, unless $n = 9$ or 10, in which case $\theta(K_9) = \theta(K_{10}) = 3$.

Bernhart and Kainen [BK79] proved that $\mathsf{bt}(K_n) = \lceil n/2 \rceil$. In fact, they proved that every convex drawing of K_n can be partitioned into $\lceil n/2 \rceil$ plane spanning paths, as illustrated in Figure **??** for K_8.

Bose *et al.* [BHRCW06] proved that every geometric drawing of K_n has thickness at most $n - \sqrt{n/12}$. It is unknown whether every geometric drawing of K_n has thickness at most $(1-\epsilon)n$. Dillencourt *et al.* [DEH00] studied the geometric thickness of K_n, and proved that

$$\lceil (n/5.646) + 0.342 \rceil \leq \overline{\theta}(K_n) \leq \lceil n/4 \rceil \ . \tag{14.2}$$

Their upper bound construction for K_8 is illustrated in Figure **??**, and it generalizes to show that for any n, $\overline{\theta}(K_n) \leq \lceil n/4 \rceil$. What is $\overline{\theta}(K_n)$? It seems likely that the answer is closer to $\lceil n/4 \rceil$ rather than to the above lower bound.

Maximum degree: Next, we consider the relationships among the three thickness parameters and the maximum degree. Recall that, a graph with maximum degree Δ is called a *degree-*Δ graph. Wessel [Wes84] and Halton [Hal91] proved independently that the thickness of a degree-Δ graph is at most $\lceil \Delta/2 \rceil$. The proof is based on the classical result of Petersen that every regular graph of even degree has a 2-factor, that is, a set of vertex disjoint cycles that together cover all the vertices. The theorem implies that the edges of a Δ-regular graph for even Δ can be partitioned into $\Delta/2$ sets of vertex disjoint cycles. Vertex disjoint cycles are planar, and thus the upper bound follows by proving that every degree-Δ graph is a subgraph of some Δ-regular graph. Sýkora *et al.* [SSV04] proved that this bound is tight.

Malitz [Mal94b] proved that there exist Δ-regular n-vertex graphs with book thickness at least $\Omega(\sqrt{\Delta} n^{1/2-1/\Delta})$. Thus, unlike thickness, book thickness is not bounded by any function of maximum degree. The proof is based on a probabilistic construction. Malitz [Mal94b] also derived an upper bound of $\mathcal{O}(\sqrt{m}) \in \mathcal{O}(\sqrt{\Delta n})$ for the book thickness, and thus the geometric thickness, of m-edge graphs.

Eppstein [Epp04a] asked whether bounded degree graphs have bounded geometric thickness. Duncan *et al.* [DEK04] gave an affirmative answer for degree-4 graphs. By the Peterson Theorem, the edges of a degree-4 graph G can be partitioned into two sets each of which induces a subgraph comprised of vertex disjoint paths and cycles in G. Duncan *et al.* [DEK04] proved that two such subgraphs can be drawn simultaneously on some planar point set using straight-line edges, thus proving that G has a geometric drawing with thickness at most two. Moreover, they provided a linear time algorithm to produce such thickness-2 geometric drawings for degree-4 graphs. In the case of degree-3 graphs, the resulting drawings fit in the $n \times n$ grid.

In a recent development, the above mentioned question of Eppstein has been answered in the negative. Barát *et al.* [BMWR3] have shown that bounded degree graphs may have unbounded geometric thickness, even approaching the square root of the number of vertices. In particular, for all $\Delta \geq 9$ there exists a Δ-regular n-vertex graph with geometric thickness $\Omega(\sqrt{\Delta} n^{1/2-4/\Delta-\epsilon})$. The proof is non-constructive and based on counting arguments. The authors have shown that there are more graphs with bounded degree than with bounded geometric thickness. To count the number of n-vertex graphs of thickness k, they considered the number of order types of n points and all the ways of connecting the points in an order type into a geometric drawing of thickness k.

Open Problem 14.6 [BMWR3] *Do degree-Δ graphs with $\Delta \in \{5, 6, 7, 8\}$ have bounded geometric thickness?*

Proper minor-closed families: Blankenship and Oporowski [Bla03, BO01] proved that all proper minor-closed families have bounded book thickness and therefore, by Equation 14.1, bounded thickness and geometric thickness. Proper minor-closed families include, for example, planar graphs, bounded genus graphs, and bounded treewidth graphs. The proof depends on Robertson and Seymour's deep structural characterization of the graphs excluding a fixed minor. As a result, the obtained bound on book thickness for graphs excluding a K_ℓ-minor is a truly huge function of ℓ.

A much better bound is known for the thickness of such families. Kostochka [Kos82] and Thomason [Tho84] proved independently that graphs excluding a K_ℓ-minor have thickness at most $\mathcal{O}(\ell \log \ell)$. Better bounds on book thickness (and thus geometric thickness) are also known for many minor-closed families. The question of book thickness of planar graphs was settled by Yannakakis [Yan86] in 1986: he proved that the book thickness of planar graphs is at most four and that there are planar graphs with book thickness matching that bound. There is some dispute over this lower bound. The construction is given in the conference version of the paper only [Yan86], where the proof is far from complete.

Endo [End97] determined that the book thickness of toroidal graphs, that is, graphs with genus one, is at most seven. Malitz [Mal94a] proved by a probabilistic argument that the book thickness of graphs with genus γ is at most $\mathcal{O}(\sqrt{\gamma})$.

Exact bounds are known for all three thickness parameters in relation to treewidth. In particular, for graphs of treewidth k the maximum thickness and the maximum geometric thickness both equal $\lceil k/2 \rceil$ [DW05]. This says that the lower bound for thickness can be matched by an upper bound, even in the more restrictive geometric setting. For graphs of treewidth k, the maximum book thickness equals k if $k \leq 2$ and equals $k+1$ if $k \geq 3$. While the lower bounds are proved in [DW05], the upper bounds on book thickness are due to Ganley and Heath [GH01].

Computational complexity: The graphs with book thickness one are precisely the outerplanar graphs [BK79], and thus can be recognized in linear time. The graphs with book thickness two are characterized as the subgraphs of planar Hamiltonian graphs [BK79], which implies that it is $\mathcal{NP}$-complete to test if $\mathsf{bt}(G) \leq 2$ [Wig82]. In fact, even determining thickness of a given book embedding is hard. Specifically, a book embedding with k pairwise crossing edges has thickness at least k, since each edge must receive a distinct color. However, the converse is not true. There exist book embeddings with no $(k+1)$ pairwise crossing edges for graphs that have thickness at least $\Omega(k \log k)$ [KK97]. Moreover, it is $\mathcal{NP}$-complete to test if a given book embedding of a graph has thickness k [GJMP80].

Testing whether a graph has thickness k is $\mathcal{NP}$-hard [Man83] even for $k = 2$. Eppstein [Epp04b] considered the problem of testing if a given geometric drawing has thickness k. For $k = 2$ the problem can be solved in polynomial time but becomes $\mathcal{NP}$-complete for $k \geq 3$. Dillencourt *et al.* [DEH00] asked what the complexity is for determining the geometric thickness of a given graph.

Open Problem 14.7 [DEH00] *Is it $\mathcal{NP}$-hard to test if the geometric thickness of a graph is k?*

We close this section with an open problem that relates book thickness and 3D grid drawings.

Open Problem 14.8 [DW04b] *Do all bipartite graphs that have book thickness three have bounded track-number?*

By studying book thickness of graph subdivisions Dujmović and Wood [DW04b] proved that an affirmative answer to this question would imply an affirmative answer to Open Problems 14.1, 14.2, and 14.3. More generally, it would imply that the queue-number is bounded by book-thickness, which is a long standing open problem [HLR92]. Since all proper minor-closed graph families have bounded book thickness [BO01], an affirmative answer to this question would further imply that all proper minor-closed graph families have linear volume 3D grid drawings.

14.5 Other (non-grid) 3D drawing conventions

3D crossing-free straight-line drawings with real coordinates: Three dimensional straight-line crossing-free graph drawings in which the vertices are allowed real coordinates have also been studied. Naturally, having a less restrictive model allows for drawings with better bounds, for example better volume bounds, in comparison to the grid model. One disadvantage to using real coordinates, however, becomes evident when a drawing is to be displayed, on a computer screen for example. Then the real vertex coordinates must be converted into integer coordinates. There are no guarantees that rounding off will maintain the correctness of the embedding.

As in the grid model, the main criterion for measuring the quality of a drawing is its volume. To make a discussion about volume meaningful, that is, to disallow arbitrary scaling, the vertices are required to lie at least unit distance apart. As noted in the introduction, a classical result of Steintz states that the triconnected planar graphs are exactly the 1-skeletons of convex polyhedra in 3D, that is, they admit 3D convex drawings. This may be considered as one of the first results in the real coordinates model. The construction, however, seems to require exponential volume in the number of vertices of a graph. The same is true for the number of bits needed to represent the coordinates of the vertices. This outlook has been greatly improved by Chrobak *et al.* [CGT96]. The technique they used to derive their results falls under the category of so-called *force directed* methods.

Forced directed methods model the graph as a physical system. For example, edges can be modeled as springs and vertices as charged particles that repel each other. A configuration where the sum of the forces on each particle is zero, that is, a local minumum of the system, gives a straight-line drawing of the graph. The famous *barycenter method* developed by Tutte [Tut60] is an example of the force directed approach. Specifically, the barycenter method takes a 3-connected plane graph G and fixes the vertices of the outer face in a convex position in the plane. The remaining vertices of G are then added one by one at the barycenter of their neighbours. The resulting system of linear equations gives coordinates for the internal vertices, and results in a 3D drawing of G where all internal faces are convex. This method can be extended to 3D.

As noted above, the best known bounds are due to Chrobak *et al.* [CGT96]. They developed a force-directed algorithm that, given an n-vertex triconnected planar graph G, outputs a 3D drawing of G with $\mathcal{O}(n)$ volume. Moreover, the vertex coordinates in the drawing can be represented by $\mathcal{O}(n \log n)$-bit rational numbers. The algorithm runs in $\mathcal{O}(M(n^{1/2}))$ time, where $M(n)$ is the time needed to multiply two $n \times n$ matrices. They also showed that if the minimum angle between two edges incident to the same vertex is required to be some fixed function of the maximum degree, then there are bounded-degree triconnected planar graphs that require $2^{\Omega(n)}$ in any 3D convex drawing.

In other results in the real coordinate model, Garg *et al.* [GTV96] proved that all graphs with bounded chromatic number can be drawn in $\mathcal{O}(n^{3/2})$ volume with constant aspect ratio and using $\mathcal{O}(\log n)$-bit rational numbers for vertex coordinates. If the number of bits is increased to $\mathcal{O}(n \log n)$, they showed that all graphs have 3D straight-line crossing-free drawings in $\mathcal{O}(n)$ volume. Their algorithms run in $\mathcal{O}(n)$ time provided that the graph coloring is given as a part of the input.

Simulated annealing techniques for generating 3D straight-line drawings have also been considered [CT96].

3D graph representations: In a *graph representation*, vertices are depicted as some set of objects and edges indicate a relationship between the objects. In the case of visibility representations, for example, there is an edge between two vertices in the graph if and only if there is a line-segment that joins the objects representing the vertices and that does not intersect any other object, that is, if the two objects are (mutually) visible. Typically, these line-segments may be required to align with an axis. In two dimensions, popular visibility representations studied are *bar*- and *rectangle visibility*. Both models are related to orthogonal drawings in the plane. Only thickness-2 graphs have such two-dimensional visibility representations, which motivates the study of 3D counterparts.

The concept generalizes naturally to three dimensions. The vertices may be disjoint 2D objects parallel to the XY-plane, and the edges may be line-segments parallel to Z-axis connecting pairs of visible objects. It is easy to see that all graphs have such a representation if the objects may be arbitrary non-convex polygons. Attention has therefore been restricted to convex polygons. For instance, K_7 has a representation with unit squares and K_8 does not, and every graphs have a representation with unit disks. Bose *et al.* [BEF$^+$98] proved that K_n has a representation with arbitrary rectangles for $n \leq 22$, while for $n \geq 56$ it does not. They also showed that all planar graphs and all complete bipartite graphs have a representation with arbitrary rectangles, but that the family of representable graphs is not closed under graph minors.

Alt *et al.* [AGW98] considered representations with arbitrary convex polygons and showed that there is no convex polygon P that would allow every complete graph to have a visibility representation by shifted copies of P. In particular, for $n > 2^{2^k}$, K_n cannot be represented by a convex k-gon. This bound has been improved by Štola[Što04], who proved that the maximum size of a complete graph with a visibility representation by copies of regular k-gon is between $k+1$ and 2^{6k}. Visibility representations with boxes have also been considered [FM99].

Kotlov *et al.* [KLV97] discovered a relationship between graph representations by touching spheres in 3D and the algebraic graph invariant μ introduced by Colin de Verdière.

Surfaces and the theory of graph minors: The field of topological graph theory studies geometric realizations of graphs in 3-space and embeddings on surfaces. Embeddings of graphs on higher surfaces are a natural generalization of embeddings in the plane.

The celebrated *graph minors theorem* of Robertson and Seymour [RS] implies that there is a finite number of forbidden minors for graphs embeddable on any given fixed surface. The Kuratowski theorem identifies the forbidden minors for the plane. The projective plane is the only other surface for which all the forbidden minors (35 of them) are known. Mohar [Moh99] gave a linear time algorithm that for any graph and any fixed surface S, either finds an embedding of the given graph in S or identifies a subgraph homeomorphic to a forbidden minor for S.

The power of the graph minors theorem can be nicely illustrated by means of the following 3D graph drawing problem. A graph is *knotless* if it has an embedding in 3D that does not contain a non-trivial knot, that is, if it has an embedding such that every cycle in the embedding bounds a disk. For example K_7 is known not to have a knotless embedding. It is easy to observe that the class of all knotless graphs is minor-closed. One algorithmic consequence of the graph minors theory is that there is a cubic time algorithm to test membership of a graph in any proper minor-closed family. Thus, remarkably, there exists a cubic time algorithm to test if a graph is knotless. This problem was not even known to be decidable before the advent of the graph minors theory. At present, however, no explicit algorithm is known, let alone a polynomial-time one, as the theorey only guarantees the existence of such an algorithm.

A related concept is that of a linkless embedding. A graph is *linkless* if it has an embedding in 3D that does not contain a pair of linked cycles, that is, two cycles in the embedding that cannot be separated by a 2-sphere embedded in 3D. For example, K_6 is known not to be linkless. Unlike the case for knotless graphs, the full characterization of linkless graphs is known. In particular, a graph is linkless if and only if it does not contain as a minor one of the six members of the Peterson family of graphs. A ΔY-*exchange* in a graph replaces a triangle by a 3-star, while a $Y\Delta$-*exchange* replaces a 3-star by a triangle. The Peterson family is comprised of the six graphs that can be obtained from K_6 by a sequence of ΔY- and $Y\Delta$-exchanges. It is also known that a graph G is linkless if and only if its Colin de Verdière invariant $\mu(G)$ is at most four. Whether knotless graphs are precisely those graphs whose Colin de Verdière invariant is at most five is an interesting open problem.

Good viewpoints: In most visualization applications, a 3D drawing of a graph will eventually be displayed as an image on some kind of 2D medium, such as a computer screen or a sheet of paper. This can be achieved by using projections. In computer graphics the most commonly used projections are the parallel and perspective projections. A 2D image, by its very nature, will necessarily contain less information than the original 3D drawing. It is therefore desirable to find *viewpoints* (the position and the direction the viewer is facing) that result in "nice" 2D images, that is, projections that preserve as much information about the 3D drawing as possible. Having an edge of the 3D drawing map to one point in the projection is lossy in that context, as is having two vertices project to the same point.

Bose *et al.* [BGRT99] developed an algorithm that, given a 3D straight-line drawing, computes an arrangement of curves that describe all bad viewpoints for that drawing. A viewpoint is bad if it maps three 3D points to the same point in the projection (vertices count as two points). Their algorithm runs in $\mathcal{O}(m^4 \log m + k)$ time, where m is the number of edges of the graph and k may be $\mathcal{O}(m^6)$ in the worst case.

The arrangement above distinguishes between bad and good viewpoints. Eades *et al.* [EHW97] studied a model with a continuous measure of goodness for a viewpoint. In particular the goodness of a viewpoint increases with distance from its nearest bad point. They also considered different definitions of bad points and developed an algorithm to compute them based on techniques of Bose *et al.* [BGRT99].

3D symmetry: Connections between symmetry and aesthetics have long been recognized. Thus displaying automorphisms of a graph as symmetries in its drawing is a very desirable feature. Drawing graphs symmetrically involves solving at least two problems. The first is to determine the symmetries (automorphisms) of a graph. The second problem is, given the graph automorphisms, to display as many of them as possible as geometric symmetries of a drawing of the graph. Symmetries in 3D can be displayed by, for example, rotation, reflection, and inversion. For a detailed account on symmetric drawings, including 3D symmetric drawings, the reader is referred to Chapter 3.

Higher dimensions: One of the basic problems in discrete geometry is determining when a graph can be realized with prescribed edge lengths in $\mathbb{R}^d$. An interesting graph invariant related to that concept is the *dimension* of a graph, introduced by Erdős *et al.* [EHT65]. It is defined as the minimum d such that the graph has a drawing in $\mathbb{R}^d$ with straight-line edges all of unit length (with possible crossings). They show, among other results, that the dimension of the complete graph K_n is $n-1$ and that the dimension of the complete bipartite graph is at most four.

A concept related to the dimensionality of a graph is that of realizability. A *realization* of a graph is a straight line "drawing" with vertices represented as points, where there is no restriction on how vertices and edges may intersect. A graph G is *d-realizable* if, given any realization of G in $\mathbb{R}^t$, there exists a realization of G with the same edge-lengths in $\mathbb{R}^d$. For example, a path is 1-realizable since its vertices can be arranged on a line with any desired edge-lengths. A tree is also 1-realizable. On the other hand, the triangle is not 1-realizable, since it has a realization in R^2 with unit distance edges but no such realization is possible in R^1. Connelly and Sloughter [BC07] proved that a graph is 1-realizable if and only if it is a forest. It is 2-realizable if and only if it has treewidth at most two, that is, if it is a series-parallel graph. They showed that a graph is 3-realizable if and only if it does not contain K_5 or an octahedral graph as a minor.

A relationship between the connectivity of graphs and higher dimensional drawings has been established [LLW88]. In particular, k-connected graphs were characterized in terms of particular convex drawings in $\mathbb{R}^{k-1}$. A forced directed method was used to derive these results.

Some other directions explored include the idea of producing 2D drawings by starting with a "nice" higher dimensional drawing of a graph and then projecting it to a plane. Higher-dimensional visibility representations with hyper-rectangles [CDH$^+$96] have also been considered.

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