Reconstructing a convex polygon from its ω-cloud⋆

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Abstract. An ω-wedge is the closed set of points contained between two rays that are emanating from a single point (the apex), and are separated by an angle $\omega < \pi$. Given a convex polygon P, we place the ω-wedge such that P is inside the wedge and both rays are tangent to P. The set of apex positions of all such placements of the ω-wedge is called the ω-*cloud* of P.

We investigate reconstructing a polygon P from its ω-cloud. Previous work on reconstructing P from probes with the ω-wedge required knowledge of the points of tangency between P and the two rays of the ω-wedge in addition to the location of the apex. Here we consider the setting where the *maximal* ω-cloud alone is given. We give two conditions under which it uniquely defines P: (i) when $\omega < \pi$ is fixed/given, or (ii) when what is known is that $\omega < \pi/2$. We show that if neither of these two conditions hold, then P may not be unique. We show that, when the uniqueness conditions hold, the polygon P can be reconstructed in $O(n)$ time with $O(1)$ working space in addition to the input, where n is the number of arcs in the input ω-cloud.

1 Introduction

"Geometric probing considers problems of determining a geometric structure or some aspect of that structure from the results of a mathematical or physical measuring device, a probe." [16, Page 1] Many probing tools have been studied in the literature such as finger probes [7], hyperplane (or line) probes [8,12], diameter probes [15], x-ray probes [9,11], histogram (or parallel x-ray) probes [13], half-plane probes [17] and composite probes [6,12] to name a few. For example, diameter probes measure the width of the polygon at a certain direction; such measurements in all directions yield the diameter function of the polygon.

A geometric probing problem can be considered as a *reconstruction* problem. Can one reconstruct an object given a set of probes? For diameter probes this is not the case: there exists a class of (curved) *orbiform* shapes such as the celebrated *Reuleaux triangle* that have the same diameter function as the circle. What is more surprising is that this is not the case even for polygonal shapes: there are uncountable families of polygons with the same diameter function, where the polygons need not be regular, nor does their

diameter function need to be constant [15]. In this paper we show, that an alternative probing device called an ω-*wedge* yields a function that is free from these drawbacks.

The ω-wedge, first studied by Bose et al. [4], is the closed set of points contained between two rays that are emanating from a single point, the *apex* of the wedge. The angle ω formed by the two rays is such that $0 < \omega < \pi$. A single probe of a convex n-gon P is *valid* when P is inside the wedge and both rays of the wedge are tangent to P, see Figure 1a.[4] A valid probe returns the coordinates of the apex and of the two points of contact between the wedge and the polygon. Using this tool, a convex n-gon can be reconstructed using between $2n - 3$ and $2n + 5$ probes, depending on the value of ω and the number of *narrow vertices* (vertices whose internal angle is at most ω) in P [4]. As an ω-wedge rotates around P, the locus of positions of the apex of the ω-wedge describes a curve called an ω-*cloud*, see Figure 1c. The ω-cloud finds applications in diverse geometric algorithms [1,2,14].

(a)   (b)   (c)
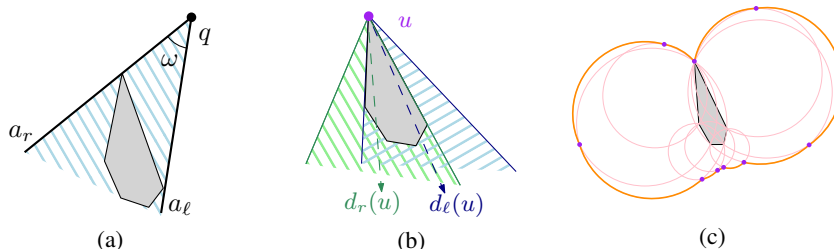
Fig. 1. A convex polygon P (filled with gray color), and (a) A minimal ω-wedge; (b) Narrow vertex u of P, wedges $W_\ell(u)$ and $W_r(u)$ (shaded with rising and falling tiling pattern, respectively) and their directions $d_\ell(u)$ and $d_r(r)$ (dashed lines); (c) The ω-cloud Ω of P: the arcs (bold curved lines), pivots (filled disk marks), and all the supporting circles (light thin curved lines).

The ω-cloud can be seen as a generalization of the diameter function introduced by Rao and Goldberg [15]. In their paper, a diameter probe consists of two parallel calipers turning around a convex object P in the plane. The function that returns the distance between these calipers as they turn around P is called a *diameter function*. Rao and Goldberg [15] show that two different convex polygons can have the same diameter function, and the function need not be a constant. This implies that recovering the exact shape and orientation of a convex n-gon given only its diameter function is not always possible and additional information is required. An ω-wedge can be seen as two non-parallel calipers turning around a convex object P in the plane. As we prove in this paper, two different convex polygons cannot have the same (non-constant) ω-cloud function, as opposed to the case of Rao and Goldberg's tool. This clearly shows the advantage of the ω-cloud against the latter tool.

Related to our probing method is also the method of Fleischer and Wang [10]. In their method, a convex n-gon P is placed inside a circle of radius 1. A camera that

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[4] In [4] probing with an ω-wedge is defined for a wider class of convex objects. Since here we focus on the objects being convex polygons, we restrict our definition accordingly.

sees only the silhouette of P can be placed anywhere on the circle. The angle with which the camera sees P together with the position of the camera is a probe of P. Let α be the largest angle of P. They prove that if no two camera tangents overlap, then $\left\lceil \frac{3\pi}{\pi-\alpha} \right\rceil$ probes are necessary and sufficient. Otherwise, approximately $\left\lceil \frac{4\pi}{\pi-\alpha} \right\rceil$ probes are sufficient. In our method, the apex of the ω-wedge that turns around P can be seen as a camera. Thus our method is a variant of theirs because instead of fixing the circle on which the camera can move, we fix the angle from which P can be seen by the camera.

In this paper, we prove that no two convex polygons have the same ω-cloud (see Theorem 2 in Section 2). We further consider a harder, but more natural problem of reconstructing a polygon from its *maximal ω-cloud*, which is a variant of an ω-cloud where the consecutive arcs with the same supporting circle are merged in one arc (see Section 2.1). We give two conditions under which the maximal ω-cloud uniquely determines P: (i) if ω with $\omega < \pi$ is fixed/given, or (ii) without fixing ω, but with a guarantee that $\omega < \pi/2$. We show that if neither of the conditions (i), (ii) hold, then P may not be unique. Finally, we show that, when conditions (i) or (ii) hold, the polygon P can be reconstructed in $O(n)$ time and $O(1)$ working space in addition to the input, where n is the number of arcs in the input ω-cloud (see Section 3).

The results in this paper differ from the ones in Bose et al. [4] in the following ways. First, we consider a different probing method, where only one piece of information is available for each probe instead of three, namely the coordinates of the apex of the ω-wedge. Second, we prove a fundamental property of the ω-cloud, that no two convex polygons have the same ω-cloud, and thus ω-clouds yield a more advantageous probing method than diameter probes. Third, we derive a list of interesting properties of the ω-cloud, which were not known before, and that allow us to provide a simple algorithm to reconstruct the polygon from its maximal ω-cloud. Note as well, that most of our results are more general than the ones from Bose et al. [4] since the results hold for any $0 < \omega < \pi$, instead of the more restricted range: $0 < \omega \leq \pi/2$. Due to space constraints, some of the proofs are omitted. They can be found in the full version of this paper [3].

2   Properties of the ω-cloud

Let P be an n-vertex convex polygon in $\mathbb{R}^2$. For any vertex v of P, let $\alpha(v)$ be the internal angle of P at v. Let ω be a fixed angle with $0 < \omega < \pi$. Consider an ω-*wedge* W; recall that it is the set of points contained between two rays a_ℓ and a_r emanating from the same point q (the *apex* of W), such that the angle between the two rays is exactly ω. See Figure 1a. We call the ray a_ℓ (resp., a_r) that bounds W from the left (resp., right) as seen from q, the *left* (resp., *right*) *arm* of W. We say that an ω-wedge W is *minimal* for P if P is contained in W and the arms of W are tangent to P. The *direction* of W is given by the bisector ray of the two arms of W. Note that for each direction, there is a unique minimal ω-wedge.

Definition 1 (ω-**cloud [5]**). *The ω-cloud of P is the locus of the apices of all minimal ω-wedges for P.*

Let Ω denote the ω-cloud of P. We define an *arc Γ* of Ω as a maximal connected portion of Ω such that for every point of Γ the corresponding minimal ω-wedge is

combinatorially the same, i.e., its left and right arm touch the same pair of vertices of P. Due to the inscribed angle theorem, an arc of the ω-cloud is a circular arc, and Ω consists of a circular sequence of circular arcs, where each pair of consecutive arcs shares an endpoint, see also Bose et al. [5] for more details. We refer to such a sequence as a circular arc sequence. The *supporting circle* of a circular arc is the circle containing the arc. The points on Ω that are the intersection of two consecutive arcs of Ω are called *pivots*. Note that if $\omega \geq \pi/2$, two consecutive arcs of the ω-cloud can lie on the same supporting circle. In this case we call the pivot separating them a *hidden pivot*. For example, points b, d, f in Figure 3a are hidden pivots for the polygon $abcdef$, when $\omega = 5/6$. The total number of pivots (including hidden ones) is between n and $2n$ [5].

A vertex v of P is called *narrow* if the angle $\alpha(v)$ is at most ω. Note that a pivot of Ω coincides with a vertex of P if and only if that vertex is narrow. In this case, we also call such pivot *narrow*. If $\alpha(v) < \omega$, we call v (the vertex or the pivot) *strictly narrow*. The portion of Ω between two points $s, t \in \Omega$, unless explicitly stated otherwise, is the portion of Ω one encounters when traversing Ω from s to t clockwise, excluding s and t. We denote this portion of Ω by Ω_{st}. The *angular measure* of an arc Γ of Ω is the angle spanned by Γ, measured from the center of its supporting circle; the angular measure of a *portion* of an arc is defined similarly. For two points s, t on Ω, the *total angular measure* of Ω from s to t, denoted as $D_\Omega(s, t)$, is the sum of the angular measures of all arcs in Ω_{st} (including the at most two non-complete arcs of Ω).

Each point x in the interior of an arc corresponds to a unique minimal ω-wedge $W(x)$ with direction $d(x)$. Let u be a pivot of Ω. If u is not strictly narrow, u also corresponds to a unique minimal ω-wedge $W(u)$ with direction $d(u)$. Otherwise, u corresponds to a closed interval of directions $[d_\ell(u), d_r(u)]$, where the angle between $d_\ell(u)$ and $d_r(u)$ equals $\omega - \alpha(u)$. See Figure 1b. Let $W_\ell(u)$ and $W_r(u)$ denote the minimal ω-wedges with apex at u and directions respectively $d_\ell(u)$ and $d_r(u)$. For points x on Ω that are not strictly narrow pivots, we define $d_r(x)$ and $d_\ell(x)$ both to be equal to $d(x)$, and $W_\ell(x)$ and $W_r(x)$ equal to $W(x)$.

The now give a crucial property of the ω-cloud that is the basis of the other properties.

Lemma 1. *Let s and t be two points on Ω such that Ω_{st} contains no narrow pivots. Then the angle between $d_r(s)$ and $d_\ell(t)$ is $D_\Omega(s, t)/2$.*

Proof. Suppose first that Ω_{st} is a single arc, see Figure 2a. The angle α between $d_r(s)$ and $d_\ell(t)$ equals the angle β between the left arms of the two minimal ω-wedges corresponding to these directions, which in turn equals angle β'. By the inscribed angle theorem, β' is half the central angle spanning the arc Ω_{st}, which, by definition of the angular measure, equals $D_\Omega(s, t)/2$.

Now suppose that Ω_{st} consists of several arcs, which are either arcs of Ω or (at most two) connected portions of those arcs. By assumption, none of the pivots separating these arcs are narrow (although s and t may be, see the definition of Ω_{st}). Consider the change of direction of the minimal ω-wedge as its apex moves from s to t. After traversing an arc with endpoints u and v, by the above observation, the direction changes by exactly $D_\Omega(u, v)/2$. Since none of the pivots on Ω_{st} are narrow, there is no change in direction as the ω-wedge passes through each pivot. Therefore the total change in direction for the ω-wedge is the sum of the changes induced by the traversed arcs, that is, $D_\Omega(s, t)/2$. □

The following are two implications of Lemma 1, and a simple fact that derives from the definition of the ω-cloud.

Corollary 1. *The sum of the angular measures of all arcs of Ω is $2(2\pi - \sum_{v \in S}(\omega - \alpha(v)))$, where S is the set of all narrow vertices of P. In particular, P has no strictly narrow vertices if and only if the sum of angular measures of the arcs of Ω is 4π.*

Corollary 2. *For any arc Γ of Ω, the angular measure of Γ is at most $2(\pi - \omega)$.*

Lemma 2. *Let u be a non-narrow pivot of Ω. Let x be the second point of intersection between the supporting circles of the two arcs of Ω adjacent to u (the first point of intersection is u itself). The minimal ω-wedge with the apex at u touches x with one of its arms.*
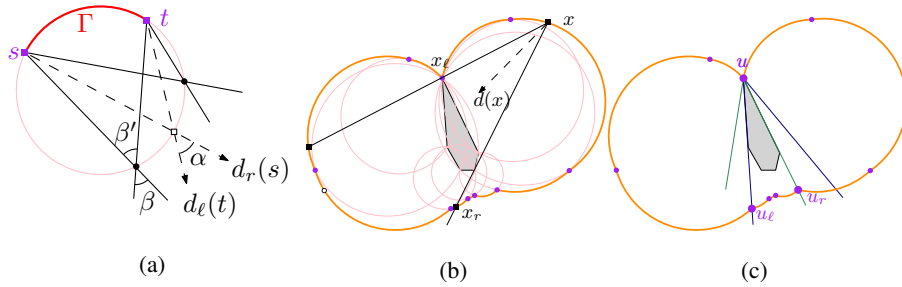

Fig. 2. (a) Illustration for the proof of Lemma 1, the case when Ω_{st} is a single arc. Vertices of P are marked as filled disks. (b) Point x in the interior of an arc of Ω, wedge $W(x)$ with its direction $d(x)$, points x_ℓ and x_r. (c) Narrow pivot u, wedges $W_\ell(u)$ and $W_r(u)$, points u_ℓ and u_r.

Characterization of narrow pivots. Let x be a point on Ω. It corresponds to two minimal ω-wedges $W_\ell(x)$, $W_r(x)$ (which coincide if x is not a narrow pivot). Consider the open ray of the right arm of $W_\ell(x)$. A small neighborhood of its origin is in the interior of the region bounded by Ω, thus the ray intersects Ω at least once. Among the points of this intersection, let x_ℓ be the one closest to x. Define the point x_r analogously for the left arm of $W_r(x)$. See Figure 2b,c.

Lemma 3. *(a) Neither $\Omega_{x_\ell x}$ nor $\Omega_{x x_r}$ contains a narrow pivot. (b) If x is a narrow pivot, then $D_\Omega(x_\ell, x) = D_\Omega(x, x_r) = 2(\pi - \omega)$. (c) If x is not narrow, then either $D_\Omega(x, x_r) = 2(\pi - \omega)$, or x_r is the first narrow pivot following x in the clockwise direction. A symmetric statement holds for x_ℓ.*

The above lemma characterizes narrow pivots in terms of the position of the corresponding minimal ω-wedges. Now we characterize them in terms of the adjacent arcs and their measures, the information that can be used by the reconstruction algorithm.

Lemma 4. *Let u be a pivot of Ω, and let v and w be the points on Ω such that $D_\Omega(v, u) = D_\Omega(u, w) = 2(\pi - \omega)$.*

(a) If pivot u is narrow, then the supporting circles of all the arcs of Ω_{vw} pass through u.
(b) Pivot u is narrow, if at least one of the following conditions is satisfied: (i) Ω_{vu} consists of a single arc; (ii) there is an arc Γ of Ω_{vu} that is not incident to u, such that the supporting circle of Γ contains u.
A symmetric statement holds for Ω_{uw}.

Observation 1. *If u is a narrow pivot followed by a non-narrow pivot, then the condition (ii) of item (b) of Lemma 4 for Ω_{uw} is satisfied, and the supporting circle of the arc following w does not pass through u.*

Proof. Let $e = uu'$ be the edge of P adjacent to u and following it in the clockwise direction. Consider the motion of the minimal ω-wedge as its apex goes from u to w. In the beginning, it rotates around u and u'. The first pivot w' the apex reaches corresponds to the change of one of those points. If u is changed, then the right arm of the wedge must be collinear with e. In this case $w' = u' = w$, and this is a narrow pivot, which contradicts the assumption of this observation. If u' is changed, then the condition (ii) of item (b) of Lemma 4 is satisfied for Ω_{uw} and the supporting circle of the arc following pivot w' is passing through u and some other vertex of P.

Let Γ' be the arc following w. Since $D_\Omega(u, w) = 2(\pi - \omega)$, the minimal ω-cloud with the apex at w has its right arm collinear with e. If the supporting circle of Γ' was passing through u, then the minimal ω-wedge $W(a)$ with the apex at any point a in the interior of Γ' would touch u with its right arm. But then the point u' is outside $W(a)$, yielding a contradiction. ☐

Characterization of hidden pivots. Recall that a pivot is called hidden if its incident arcs have the same supporting circle.

Lemma 5. *Let u be a hidden pivot of Ω, let Γ_ℓ and Γ_r be the two arcs of Ω incident to u, and let v and w be the other endpoints of Γ_ℓ and Γ_r, respectively. Then v, u, and w are all narrow and each of the arcs Γ_ℓ, Γ_r has angular measure $2(\pi - \omega)$.*

Proof. Since u is a hidden pivot, Γ_ℓ and Γ_r are supported by the same circle C. Consider the minimal ω-wedge as its apex traverses Γ_ℓ. Its arms are touching two vertices of P, both lying on C. Since u is a hidden pivot, the vertex touched by the left arm of the wedge is u (otherwise, there would be no possibility to switch from Γ_ℓ to Γ_r at point u). Let a be the vertex of P touched by the right arm of the wedge. When the apex of the wedge reaches u, the wedge becomes $W_\ell(u)$, and its right arm is passing through a and u. If polygon P had a vertex between a and u, that vertex must lie outside W_ℓ, which is impossible. Thus a is actually v. Therefore, $D_\Omega(v, u) = D_\Omega(a, u)$, which by Lemma 3b equals $2(\pi - \omega)$. A symmetric argument for u, v, and Γ_r completes the proof. ☐

We proceed with the main result of this section.

Theorem 2. *For a circular arc sequence Ω, there is at most one convex polygon P and one angle ω with $0 < \omega < \pi$ such that Ω is the ω-cloud of P.*

2.1   Maximal ω-cloud

Given the ω-cloud Ω of P, let the *maximal ω-cloud* of P, denoted Ω^*, be the result of merging all the pairs of consecutive arcs in Ω that have same supporting circle; equivalently Ω^* can be seen as the result of removing all the hidden pivots from Ω.

The maximal ω-cloud is a natural modification of the ω-cloud for the reconstruction task, see Section 3; it reflects the situation when as an input we are given a locus of the apices of all the minimal ω-wedges without any additional information, i.e., without the coordinates of the hidden pivots. The following two statements are corollaries of Lemma 5.

Lemma 6. *The maximal ω-cloud of P is a circle C if and only if $k = \pi/(\pi - \omega)$ is an integer and P is a regular k-gon inscribed in C.*

Proof. Suppose the maximal ω-cloud Ω^* of P is a circle C. Since all arcs of Ω^* of P are supported by the circle C, all the pivots of Ω are hidden. Applying Lemma 5 to each pivot, we obtain that the pivots of Ω are exactly the vertices of P, and each arc has measure $2(\pi - \omega)$. Therefore the number of vertices of P is $2\pi/(2(\pi - \omega))$, and the claim follows. The other direction of the claim directly follows from the definition of an ω-cloud. $\square$

Lemma 7. *An arc of Ω^* is greater than $2(\pi - \omega)$ if and only if this arc is a concatenation of at least two co-circular arcs of Ω separated by hidden pivots, each of measure $2(\pi - \omega)$.*

Lemmas 6 and 7 together with Theorem 2 imply a similar uniqueness result for Ω^*:

Theorem 3. *Given an angle ω, and a circular arc sequence Ω^* of at least two arcs, there is at most one convex polygon P such that Ω^* is the maximal ω-cloud of P.*

Another uniqueness result, which is useful for our reconstruction algorithm, follows from Theorem 2 and the fact that for $\omega < \pi/2$ the ω-cloud and the maximal ω-cloud coincide.

Corollary 3 (of Theorem 2). *For a circular arc sequence Ω, there is at most one convex polygon P and one angle $\omega < \pi/2$ such that Ω is the maximal ω-cloud of P.*

We now show that the condition that $\omega < \pi/2$ is necessary for the above statement.

Proposition 1. *There is a circular arc sequence such that it is the maximal ω-cloud of P and the maximal ω'-cloud of P' for distinct angles ω, ω' and polygons P, P'.*

Proof. Let P be a triangle with vertices a, c, e, such that $\angle ace = \pi/3$, $\angle cea = 16/45\pi$, and $\angle cae = 14/45\pi$. See Figure 3a. Let $\omega = 2/3\pi$, and let Ω be the ω-cloud of P; it consists of three circular arcs Γ_0, Γ_1, and Γ_2; note that the supporting circles of these arcs are pairwise distinct. Since Ω has no hidden pivots, Ω is also the maximal ω-cloud of P. Let b (resp., d and f) be the midpoint of Γ_0 (resp., of Γ_1 and of Γ_2), and let P' be the hexagon $abcdef$. Then $\angle abc = \angle cde = \angle efa = \angle bcd = 2/3\pi$, $\angle fab = 29/45\pi$, and $\angle fab = 29/45\pi$. Let $\omega' = 5/6\pi$, and note that all the angles at the vertices of P' are smaller than ω'. It is easy to verify that Ω is the maximal ω'-cloud of P'. $\square$

Remark 1. The example in the proof of Proposition 1 can be continued by subdividing each arc in half and considering $11/12\pi$-cloud for the obtained 12-gon, and so on.
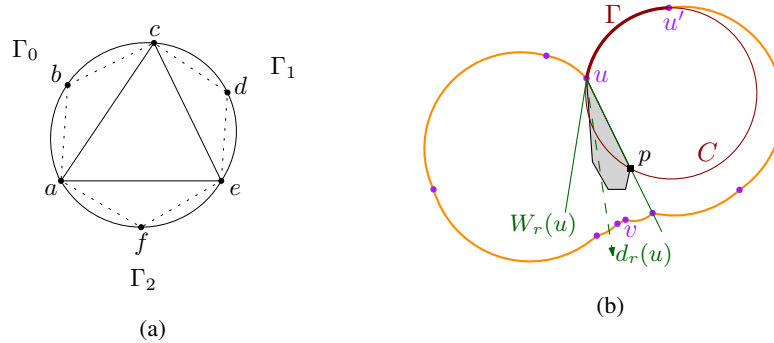
Fig. 3. (a) A circular arc sequence of 3 arcs, that is the maximal $2/3\pi-$cloud for triangle ace and the maximal $5/6-$cloud for hexagon $abcdef$. (b) Illustration for the proof of Lemma 8.

3   Reconstructing P from its maximal ω-cloud

In this section we let Ω^* be a circular arc sequence. Our goal is to reconstruct the convex polygon P for which Ω^* is the maximal ω-cloud for some angle ω, or determine that no such polygon exists.

Note that if Ω^* is a single arc, i.e., it is a circle C, then P is not unique. By Lemma 6, it is a regular $\pi/(\pi - \omega)$-gon inscribed in C. The position of its vertices on C is impossible to identify given only Ω^* and ω. Therefore we assume that Ω^* has at least two arcs.

First, in Section 3.1, we consider ω to be given, and such that $0 < \omega < \pi$; in this case the maximal ω-cloud of P may differ from its ω-cloud. Afterwards, in Section 3.2, we consider the setting where ω is not known, but it is known that $0 < \omega < \pi/2$; in this case the two variations of the ω-cloud coincide, and we use Ω instead of Ω^* to denote the input.

3.1   An ω-aware reconstruction algorithm

Here we assume that ω is given, and that $0 < \omega < \pi$. The main difficulty in the reconstruction task is caused by strictly narrow vertices, as the turn of the minimal ω-wedge at those vertices is not reflected in the ω-cloud, see Corollary 1. In our reconstruction algorithm, we first find all the non-hidden strictly narrow pivots of Ω^*, and then treat the connected portions of Ω^* between those pivots separately. Lemma 8 gives a procedure to process such a portion.

For two points u and v on Ω^*, let P_{uv} be the union of the edges and vertices of P touched by the arms of the minimal ω-wedge as its apex traverses Ω^*_{uv}. Since Ω^*_{uv} is connected, when the apex of the wedge traverses Ω^*_{uv}, for each of its two arms, the point of contact with P traverses a connected portion of P; thus P_{uv} consists of at most two connected portions of P. Note that it is possible that one of the portions is a single vertex.

Lemma 8. *Given a portion Ω_{uv}^* of Ω^* that does not contain any strictly narrow pivots, and the direction $d_r(u)$ of the rightmost minimal ω-wedge $W_r(u)$ of u, the portion P_{uv} of P that corresponds to Ω_{uv}^* can be reconstructed in time linear in the number of arcs in Ω_{uv}^*. The procedure requires $O(1)$ working space in addition to the input.*

Proof. Let Γ be the arc of Ω_{uv}^* incident to u, let u' be the other endpoint of Γ, and let C be the supporting circle of Γ. See Figure 3b. By knowing the value of ω and the direction $d_r(u)$, we determine the wedge $W_r(u)$. The intersection between the wedge $W_r(u)$ and the circle C determines the two vertices of P touched by the minimal ω-wedge as its apex traverses arc Γ. In Figure 3b, these two points are u and p. The direction of the leftmost minimal ω-wedge at u', $W_\ell(u')$, is $d_r(u) + D_\Omega(u, u')/2$ due to Lemma 1. If u' is inside Ω_{uv}^*, then u' is not a strictly narrow vertex, and thus there is a unique minimal ω-wedge $W(u')$ at u', $W(u') = W_\ell(u')$. Therefore, for each arc of Ω_{uv}^* we find the pair of vertices of P that induces that arc. Moreover, by visiting the pivots of Ω_{uv}^* one by one, we find the vertices of each of the two chains of P_{uv} ordered clockwise. To avoid double-reporting vertices of P, we keep the startpoints of the two chains, and whenever one chain reaches the startpoint of the other one, we stop reporting the points of the former chain.

This procedure visits the pivots of Ω_{uv}^* one by one, and performs $O(1)$ operations at each pivot, namely, finding the intersection between a given wedge and a given circle. No additional information needs to be stored. $\square$

Reconstruction algorithm. As an input, we are given an angle ω, $0 < \omega < \pi$, and a circular arc sequence Ω^* which is not a single circle. We now describe an algorithm to check if Ω^* is the ω-cloud of some convex polygon P, and to return P if this is the case. It consists of two passes through Ω^*, which are detailed below. During the first pass we compute a list S of all strictly narrow vertices of P that are not hidden pivots. With each such vertex u, we store the supporting lines of the two edges of P incident to u. In the second pass we use this list to reconstruct P.

First pass. We iterate through the pivots of Ω^* (recall that these are not hidden by the definition of Ω^*). For the currently processed pivot u, we maintain the point v on Ω^* such that $D_\Omega(v, u) = 2(\pi - \omega)$. If pivot u is narrow, we jump to the point on Ω^* at the distance $2(\pi - \omega)$ from u. Moreover, if u is strictly narrow, we add u to the list S. If u is not narrow, we process the next pivot of Ω^*. We now give the details.

Let Γ be the arc of Ω^* incident to u and following it in clockwise direction. Let Γ_r be the arc following Γ, and C_r be the supporting circle of Γ_r. We consider several cases depending on the angular measure $|\Gamma|$ of Γ:

(a) $|\Gamma| < 2(\pi - \omega)$. See Figure 4.
  (i) Circle C_r passes through u, see Figure 4a. Then u is narrow by Lemma 4b. By tracing Ω^*, find the point w on it such that $D_\Omega(u, w) = 2(\pi - \omega)$. In case u is strictly narrow (i.e., $\angle vuw < \omega$), add u to the list S with the lines through vu and uw (these are the intended lines due to Lemma 3). Set v to be u, and u to be w (regardless the later condition).
  (ii) Circle C_r does not pass through u, see Figure 4b. Then u is not narrow by Lemma 4a. Set u to be the other endpoint of Γ, and update v accordingly.
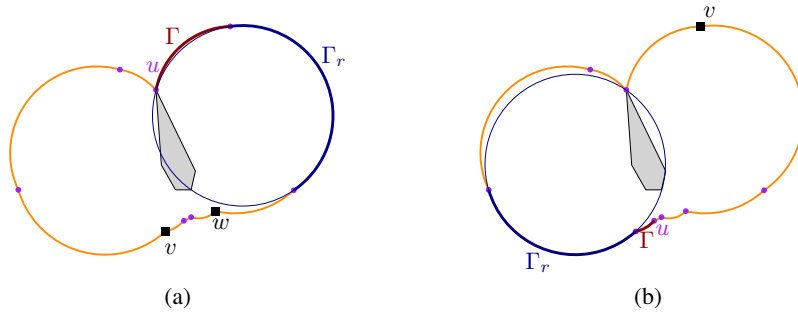
Fig. 4. First pass of the ω-aware algorithm: pivot u is (a) strictly narrow, (b) non-narrow

(b) $|\Gamma| = 2(\pi - \omega)$. Then u is narrow by Lemma 4b. Let w be the other endpoint of Γ. Update S, v, and u as in item a(i).

(c) $|\Gamma| = 2t(\pi - \omega)$ for some integer $t > 1$. Then Γ is in fact multiple arcs separated by hidden pivots, see Lemma 5 and Corollary 2. Let p be the other endpoint of Γ. Let w and w' be the points on Γ such that $D_\Omega(u, w) = 2(\pi - \omega)$ and $D_\Omega(w', p) = 2(\pi - \omega)$. Update S as in item a(i). Set u to be p, and v to be w'.

(d) Otherwise, stop and report that Ω^* is not the maximal ω-cloud of any polygon.

Note that before starting the above procedure, we need to find the starting positions v and u. We choose v arbitrarily and traverse Ω^* until we reach the corresponding position of u (i.e., $D_\Omega(v, u) = 2(\pi - \omega)$). Since we already have traversed the portion of Ω^* between v and u, in order to perform exactly one pass through Ω^*, we will have to finish the above procedure as soon as the pointer u has reached the initial position of v (not the initial position of u). However, this is still enough for creating the complete list S. Indeed, by Lemma 3b, there is at most one narrow pivot between v and u. There is no such narrow pivot if and only if the left arm of the minimal ω-wedge $W_r(v)$ and the right arm of $W_\ell(u)$ coincide, and in this case there is nothing to add to S. If there is such a narrow pivot w, then neither u nor v is narrow. By Lemma 3c, both the left arm of the wedge $W(v)$ and the right arm of $W(u)$ pass through w. Thus w can be found as their intersection.

Second pass. In case list S is empty, we apply the procedure of Lemma 8 to the whole Ω^*. In particular, as both the start and the endpoint, we take the point x with which we completed the first pass of the algorithm; the point x' such that $D_\Omega(x', x) = 2(\pi - \omega)$ is already known from the first pass. Then $d_r(x) = d(x)$ is the direction of the minimal ω-wedge with the apex at x and the right arm passing through x'.

Suppose now that list S contains k vertices. They subdivide Ω^* into k connected portions that are free from strictly narrow non-hidden pivots. Each portion is treated as follows:

– If it is a single maximal arc of measure $2t(\pi - \omega)$, we simply separate it by $t - 1$ equidistant points, and those points are exactly the vertices of the portion of P corresponding to the considered component of Ω^*, see Lemma 5.

– Otherwise it is a portion free from any strictly narrow pivots. We process this portion by the procedure of Lemma 8.

Correctness and complexity of this algorithm are analyzed in the following theorem.

Theorem 4. *Given an angle ω such that $0 < \omega < \pi$, and a circular arc sequence Ω^* of n arcs with $n > 1$, there is an algorithm to check if Ω^* is the maximal ω-cloud of some convex polygon P, and to return P if this is the case. The algorithm works in $O(n)$ time, making two passes through the input, and it uses $O(1)$ working space. The constants in big-Oh depend only on ω.*

Proof. Consider the first pass of the algorithm. Cases a(i), b, and c are exactly the cases where u is a narrow pivot, see Lemmas 4 and 5. The case d corresponds to an impossible situation by Lemma 5 and Corollary 2. In case a(i), we update u to be w, thus we are not processing the pivots between u and w; Lemma 3 guarantees we do not miss any narrow pivot. Correctness of the second pass is implied by Lemmas 5 and 3a.

Each pivot of Ω^* is visited exactly once during the first pass (it is either checked as the pivot u, traversed in the case a(i), or skipped in the case c). The second pass as well visits each pivot once. The time spent during each such visit is $O(1 + t)$, where t is the number of hidden pivots on the maximal arc adjacent to the visited pivot. The storage required by the algorithm is the storage required for the list S, i.e., it is $O(k)$.

Observe finally that by Lemma 3, for any pair u, v of narrow pivots of Ω^*, $D_\Omega(u, v) \geq 2(\pi - \omega)$. By Corollary 1 the total angular measure of the arcs of Ω^* is at most 4π. Thus the total number of narrow pivots of Ω^* (including the hidden pivots) is at most $\lfloor 2\pi/(\pi - \omega) \rfloor$, which is a constant if ω is fixed. This completes the proof. $\qquad\square$

Remark 2. It is not difficult to modify the above algorithm to work for the easier setting, where the input is the proper ω-cloud of P instead of its maximal version.

3.2   An ω-oblivious reconstruction algorithm

In this section we assume that $\omega < \pi/2$. Thus there are no hidden pivots in the ω-cloud of P, and therefore the input sequence equals the (not maximal) ω-cloud Ω. We now give the reconstruction algorithm, which is the main result of this section.

Theorem 5. *Given a circular arc sequence Ω, there is an algorithm that finds the convex polygon P such that Ω is the ω-cloud of P for some angle ω with $0 < \omega < \pi/2$, if such a polygon exists. Otherwise, it reports that such a polygon does not exist. The algorithm works in $O(n)$ time, making two passes through the input, and it uses $O(1)$ working space.*

Proof. If Ω is a single arc, i.e., it is a circle, we return the negative answer, as a circle cannot be an ω-cloud for $\omega < \pi/2$ by Lemma 6. Below we assume that Ω has at least two arcs.

If the total angular measure of Ω is 4π, then no pivot is strictly narrow due to Corollary 1. Starting from any pivot u of Ω and applying Lemma 2, we find the placement of one of the arms of the minimal ω-wedge with the apex u. Since no pivot is strictly narrow, Lemma 3c determines whether it is the right or the left arm. This gives us the angle and the placement of the wedge. We can perform the procedure of Lemma 8 to reconstruct the polygon.

If the total angular measure of Ω is less than 4π, there must be a strictly narrow pivot. We perform a pass through Ω to find a maximal sequence of arcs whose supporting circles contain the same pivot u, and such that the sequence starts at u. If we found such sequence of at least two arcs, the distance between u and the endpoint of this sequence is $2(\pi - \omega)$ due to Lemma 4 and Observation 1. This determines the value of ω. We run the ω-aware reconstruction procedure of Section 3.1. If there is no such sequence of at least two arcs, then by Observation 1 all pivots must be narrow, but since $\omega < \pi/2$, the polygon P in that case must be a line segment. Whether Ω is an ω-cloud of a line segment for some ω can be checked in one pass through Ω.

We can combine finding the above maximal sequence with counting the total angular measure of Ω in one pass; this pass requires $O(1)$ working storage. The claim then follows from the Theorem 4. □

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