Optimal Bounds on Theta-Graphs: More is not Always Better

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Abstract

We present tight upper and lower bounds on the spanning ratio of a large family of θ-graphs. We show that θ-graphs with $4k+2$ cones ($k \geq 1$ and integer) have a spanning ratio of $1 + 2\sin(\theta/2)$, where θ is $2\pi/(4k+2)$. We also show that θ-graphs with $4k+4$ cones have spanning ratio at least $1 + 2\tan(\theta/2) + 2\tan^2(\theta/2)$, where θ is $2\pi/(4k+4)$. This is somewhat surprising since, for equal values of k, the spanning ratio of θ-graphs with $4k+4$ cones is greater than that of θ-graphs with $4k+2$ cones, showing that increasing the number of cones can make the spanning ratio worse.

1 Introduction

In a weighted graph G, let the distance $\delta_G(u, v)$ between two vertices u and v be the length of the shortest path between u and v in G. A subgraph H of G is a *t-spanner* of G if for all pairs of vertices u and v, $\delta_H(u, v) \leq t \cdot \delta_G(u, v), t \geq 1$. The *spanning ratio* of H is the smallest t for which H is a t-spanner. The graph G is referred to as the *underlying graph*.

We consider the situation where the underlying graph G is a straightline embedding of the complete graph on a set of n points in the plane denoted by K_n, with the weight of an edge (u, v) being the Euclidean distance $|uv|$ between u and v. A spanner of such a graph is called a *geometric spanner*. We look at a specific type of geometric spanner: θ-graphs.

Introduced independently by Clarkson [4] and Keil [5], θ-graphs are constructed as follows (a more precise definition follows in the next section): for each vertex u, we partition the plane into m disjoint cones with apex u, each having aperture $\theta = 2\pi/m$. When m cones are used, we denote the resulting θ-graph as θ_m. The θ-graph is constructed by, for each cone with apex u, connecting u to the vertex v whose projection along the bisector of the cone is closest. Ruppert and Seidel [6] showed that the spanning ratio of these graphs is at most $1/(1 - 2\sin(\theta/2))$, when $\theta < \pi/3$, i.e. there are at least seven cones.

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Recently, Bonichon *et al.* [1] showed that the θ_6-graph has spanning ratio 2. This was done by dividing the cones into two sets, positive and negative cones, such that each positive cone is adjacent to two negative cones and vice versa. It was shown that when edges are added only in the positive cones, in which case the graph is called the half-θ_6-graph, the resulting graph is equivalent to the TD-Delaunay triangulation (the Delaunay triangulation where the empty region is an equilateral triangle) whose spanning ratio is 2 as shown by Chew [3]. An alternative, inductive proof of the spanning ratio of the θ_6-graph was presented by Bose *et al.* [2].

Tight bounds on spanning ratios are notoriously hard to obtain. The standard Delaunay triangulation (where the empty region is a circle) is a good example. It has been studied for over 20 years and the upper and lower bounds still do not match. Also, even though it was introduced about 25 years ago, the spanning ratio of the θ_6-graph has only recently been shown to be finite and tight, making it the first and, until now, only θ-graph for which tight bounds are known.

In this paper, we generalize the results from Bose *et al.* [2]. We look at two families of θ-graphs: the $\theta_{(4k+2)}$-graph and the $\theta_{(4k+4)}$-graph, where k is an integer and at least 1. We show that the $\theta_{(4k+2)}$-graph has a tight spanning ratio of $1 + 2\sin(\theta/2)$ and that the $\theta_{(4k+4)}$-graph has a strictly larger spanning ratio of at least $1 + 2\tan(\theta/2) + 2\tan^2(\theta/2)$, for their respective values of θ.

2 Preliminaries

Let a *cone* C be the region in the plane between two rays originating from the same point (referred to as the apex of the cone). When constructing a θ_m-graph, for each vertex u of K_n consider the rays originating from u with the angle between consecutive rays being $\theta = 2\pi/m$. Each pair of consecutive rays defines a cone. The cones are oriented such that the bisector of some cone coincides with the vertical line through u.

The θ_m-graph is constructed as follows: for each cone C of each vertex u, add an edge from u to the closest vertex in that cone, where distance is measured along the bisector of the cone. More formally, we add an edge between two vertices u and v if $v \in C$ and for all vertices $w \in C$ $(v \neq w)$, $|uv'| \leq |uw'|$, where v' and w' denote the orthogonal projection of v and w on the bisector of C.

For ease of exposition, we only consider point sets in general position: no two vertices lie on a line parallel to one of the rays that define the cones and no two vertices lie on a line perpendicular to the bisector of one of the cones. This implies that each vertex adds at most one edge per cone to the graph.

Given a vertex w in cone C of vertex u, we define the *canonical triangle* T_{uw} to be the triangle defined by the borders of C and the line through w perpendicular to the bisector of C. We use m to denote the midpoint of the side of T_{uw} opposite u and α to denote the unsigned angle between uw and um. See Figure 1. Note that for any pair of vertices u and w, there exist two canonical triangles: T_{uw} and T_{wu}.

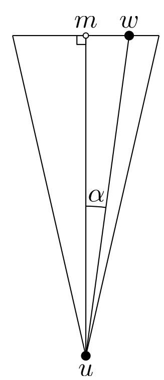

Figure 1: The canonical triangle T_{uw}

3 Spanning Ratio of the $\theta_{(4k+2)}$-Graph

In this section, we give matching upper and lower bounds on the spanning ratio of the $\theta_{(4k+2)}$-graph, for any integer $k \geq 1$. The proof is a generalization of the proof given by Bose *et al.* [2]. We first show that the $\theta_{(4k+2)}$-graph has a very nice geometric property:

Lemma 1 *Any line perpendicular to the bisector of a cone is parallel to the boundary of some cone.*

Proof. The angle between the bisector of a cone and the boundary of that cone is $\theta/2$ and the angle between the bisector and the line perpendicular to this bisector is $\pi/2 = ((2k+1)/2) \cdot \theta$. Thus the angle between the line perpendicular to the bisector and the boundary of the cone is $2\pi - \theta/2 - ((2k+1)/2) \cdot \theta = k \cdot \theta$. Since a cone boundary is placed at every multiple of θ, the line perpendicular to the bisector is parallel to the boundary of some cone. □

This property implicitly helps when bounding the spanning ratio of the $\theta_{(4k+2)}$-graph. However, before deriving this bound, we first prove a useful geometric lemma.

Lemma 2 *Given a convex quadrilateral $abcd$ such that no three of its vertices lie on a line, $\angle abc = \angle adc$, $\angle bad \leq \angle bcd$, and $\angle bad \leq 2 \cdot \angle bac$. It holds that $|ad| + |dc| \leq |ab| + |bc|$.*

Proof. Since $\angle bad \leq 2 \cdot \angle bac$, the bisector of $\angle bad$ intersects bc. Let x be this intersection. Let y be the intersection of ad and the line through x, parallel to cd. Since $\angle bad \leq \angle bcd$, the line through d parallel to bc intersects xy. Let z be this intersection. See Figure 2. These definitions imply that $|zd| = |xc|$ and $|zx| = |dc|$.

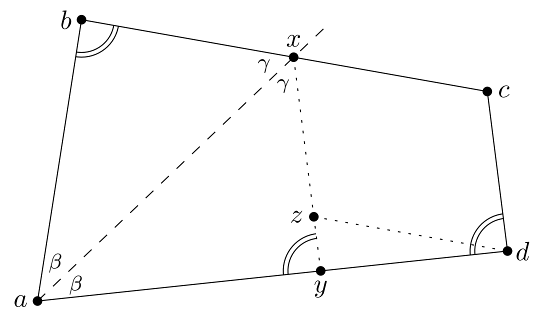

Figure 2: Quadrilateral $abcd$

Since $\angle bax = \angle yax$ and $\angle abx = \angle adc = \angle ayx$, we have that $\angle bxa = \angle yxa$. Since ax is part of both triangle abx and triangle ayx, the law of sines implies that $|ab| = |ay|$ and $|bx| = |yx|$. We now rewrite $|ad| + |dc|$ and $|ab| + |bc|$:

$$\begin{aligned} |ad| + |dc| &= |ay| + |yd| + |dc| \\ &= |ay| + |yd| + |zx| \end{aligned}$$

$$\begin{aligned} |ab| + |bc| &= |ab| + |bx| + |xc| \\ &= |ay| + |yx| + |xc| \\ &= |ay| + |yz| + |zx| + |zd| \end{aligned}$$

Therefore $|ad| + |dc| \leq |ab| + |bc|$ if and only if $|yd| \leq |yz| + |zd|$, which follows from the triangle inequality. □

Theorem 3 *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path in the $\theta_{(4k+2)}$-graph of length at most*

$$\left(\left(\frac{1 + \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \right) \cdot \cos\alpha + \sin\alpha \right) \cdot |uw|.$$

Proof. We prove the theorem by induction on the area of T_{uw} (formally, induction on the rank, when ordered by area, of the canonical triangles for all pairs of vertices). Let a and b be the upper left and right corners of T_{uw}, let p and q be the intersections of T_{uw} and the lower boundaries of the uppermost cones of w that intersect T_{uw}, and let x and y be the left and right intersections of T_{uw} and the boundaries of the cone of w that contains u. See Figure 3.

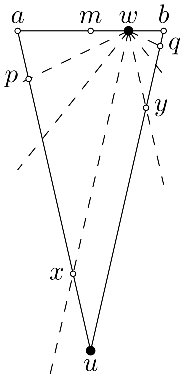

Figure 3: The canonical triangle T_{uw} with a, b, p, q, x, and y being the various intersections of its sides

Our inductive hypothesis is the following, where $\delta(u, w)$ denotes the length of the shortest path from u to w in the $\theta_{(4k+2)}$-graph:

- If axw is empty, then $\delta(u, w) \leq |ub| + |bw|$.
- If byw is empty, then $\delta(u, w) \leq |ua| + |aw|$.
- If neither axw nor byw is empty, then $\delta(u, w) \leq \max\{|ua| + |aw|, |ub| + |bw|\}$.

We first show that this induction hypothesis implies the theorem. Basic trigonometry gives us the following equalities: $|um| = |uw| \cdot \cos\alpha$, $|mw| = |uw| \cdot \sin\alpha$, $|am| = |bm| = |uw| \cdot \cos\alpha \cdot \tan(\theta/2)$, and $|ua| = |ub| = |uw| \cdot \cos\alpha / \cos(\theta/2)$. Thus the induction hypothesis gives that $\delta(u, w)$ is at most $|ua| + |am| + |mw| = |uw| \cdot (((1 + \sin(\theta/2))/\cos(\theta/2)) \cdot \cos\alpha + \sin\alpha)$.

Base case: T_{uw} has rank 1. Since the triangle is a smallest triangle, w is the closest vertex to u in that cone. Hence the edge (u, w) is part of the $\theta_{(4k+2)}$-graph, and $\delta(u, w) = |uw|$. From the triangle inequality, we have $|uw| \leq \min\{|ua| + |aw|, |ub| + |bw|\}$, so the induction hypothesis holds.

Induction step: We assume that the induction hypothesis holds for all pairs of vertices with canonical triangles of rank up to i. Let T_{uw} be a canonical triangle of rank $i + 1$.

If (u, w) is an edge in the $\theta_{(4k+2)}$-graph, the induction hypothesis follows by the same argument as in the base case. If there is no edge between u and w, let v be the vertex closest to u in the cone of u that contains w, and let a' and b' be the upper left and right corners of T_{uv}. See Figure 4. By definition, $\delta(u, w) \leq |uv| + \delta(v, w)$, and by the triangle inequality, $|uv| \leq \min\{|ua'| + |a'v|, |ub'| + |b'v|\}$.

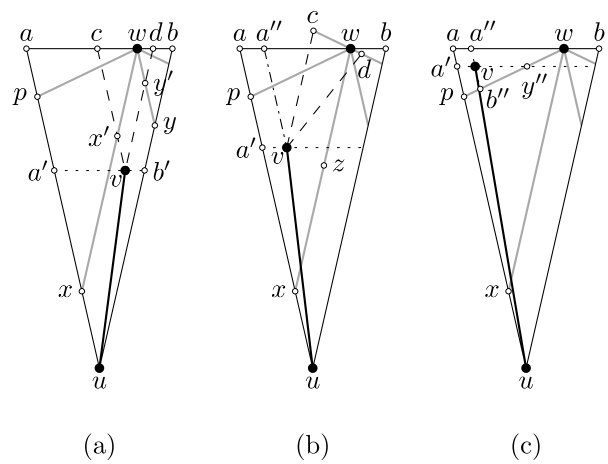

Figure 4: The three cases: (a) v lies in $uxwy$, (b) v lies in xpw, (c) v lies in paw

We perform a case analysis based on the location of v: (a) v lies in $uxwy$, (b) v lies in xpw, (c) v lies in paw, (d) v lies in yqw, and (e) v lies in qbw. Case (d) is analogous to Case (b) and Case (e) is analogous to Case (c), so we only discuss the first three cases.

Case (a): Vertex v lies in $uxwy$. Let c and d be the upper left and right corners of T_{vw}, and let x' and y' be the left and right intersections of T_{vw} and the boundaries of the cone of w that contains v. See Figure 4a. Since T_{vw} has smaller area than T_{uw}, we apply the inductive hypothesis on T_{vw}. Our task is to prove all three statements of the inductive hypothesis for T_{uw}.

1. If axw is empty, then $cx'w$ is also empty, so by induction $\delta(v, w) \leq |vd| + |dw|$. Since v, d, b, and b' form a parallelogram, we have:

$$\begin{aligned} \delta(u, w) &\leq |uv| + \delta(v, w) \\ &\leq |ub'| + |b'v| + |vd| + |dw| \\ &= |ub| + |bw|, \end{aligned}$$

which proves the first statement of the induction hypothesis.

2. If byw is empty, an analogous argument proves the second statement of the induction hypothesis.

3. If neither axw nor byw is empty, by induction we have $\delta(v, w) \leq \max\{|vc| + |cw|, |vd| + |dw|\}$. Assume, without loss of generality, that the maximum of the right hand side is attained by its second argument $|vd| + |dw|$ (the other case is analogous).

Since vertices v, d, b, and b' form a parallelogram, we have that:

$$\begin{aligned}\delta(u,w) &\leq |uv| + \delta(v,w)\\ &\leq |ub'| + |b'v| + |vd| + |dw|\\ &\leq |ub| + |bw|\\ &\leq \max\{|ua| + |aw|, |ub| + |bw|\},\end{aligned}$$

which proves the third statement of the induction hypothesis.

Case (b): Vertex v lies in xpw. Since v lies in axw, the first statement in the induction hypothesis for T_{uw} is vacuously true. It remains to prove the second and third statement in the induction hypothesis. Let c and d be the upper and lower right corners of T_{vw}, and let a'' be the intersection of aw and the line through v, parallel to ua. See Figure 4b. Since T_{vw} is smaller than T_{uw}, by induction we have $\delta(v,w) \leq \max\{|vc| + |cw|, |vd| + |dw|\}$. We perform a case analysis based on this: (i) $\delta(v,w) \leq |vd| + |dw|$, (ii) $\delta(v,w) \leq |vc| + |cw|$.

Case (i): Since $\angle va''w$ and $\angle vdw$ are both the angle between the boundary of a cone and the line perpendicular to the bisector of that cone, we have $\angle va''w = \angle vdw = k \cdot \theta$. Also, we have that $\angle a''vd \leq \angle a''wd$, since $\angle a''vd \leq k \cdot \theta$ and

$$\begin{aligned}\angle a''wd &= 2\pi - \angle va''w - \angle vdw - \angle a''vd\\ &\geq (4k+2)\cdot\theta - 3k\cdot\theta\\ &= (k+2)\cdot\theta\end{aligned}$$

Furthermore, since $\angle a''vw > \angle a''vc \geq \theta$ and $\angle a''vd = \angle a''vc + \theta \leq 2 \cdot \angle a''vc$, we have that $\angle a''vd < 2 \cdot \angle a''vw$.

Hence we can apply Lemma 2 to quadrilateral $va''wd$, which gives us that $|vd| + |dw| \leq |va''| + |a''w|$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we have that $\delta(u,w) \leq |ua| + |aw|$, proving the second and third statement in the induction hypothesis for T_{uw}.

Case (ii): Let z be the lower corner of T_{wv}. Since $vcwz$ form a parallelogram, we know that $|vc| + |cw| = |wz| + |zv|$. We now look at quadrilateral $wzva''$. Analogous to Case (i), we have that $\angle wzv = \angle wa''v = k \cdot \theta$, $\angle a''wz \leq \angle a''vz$, and $\angle a''wz < 2 \cdot \angle a''wv$. Hence we can apply Lemma 2 to quadrilateral $wzva''$, which gives us that $|wz| + |zv| \leq |va''| + |a''w|$, proving the second and third statement in the induction hypothesis for T_{uw}.

Case (c): Vertex v lies in paw. Since v lies in axw, the first statement in the induction hypothesis for T_{uw} is vacuously true. It remains to prove the second and third statement in the induction hypothesis. Let a'' and b'' be the upper and lower left corners of T_{wv}, and let y'' be the intersection of T_{wv} and the lower boundary of the cone of v that contains w. See Figure 4c. Note that y'' is also the right intersection of T_{uv} and T_{wv}. Since v is the closest vertex to u, T_{uv} is empty. Hence, $b''y''v$ is empty. Since T_{wv} is smaller than T_{uw}, we can apply induction on it. As $b''y''v$ is empty, the first statement of the induction hypothesis for T_{wv} gives $\delta(v,w) \leq |va''| + |a''w|$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we have that $\delta(u,w) \leq |ua| + |aw|$, proving the second and third statement in the induction hypothesis for T_{uw}. □

Since $((1+\sin(\theta/2))/\cos(\theta/2)) \cdot \cos\alpha + \sin\alpha$ is increasing for $\alpha \in [0, \theta/2]$, for $\theta \leq \pi/3$, it is maximized when $\alpha = \theta/2$, and we obtain the following corollary:

Corollary 4 *The $\theta_{(4k+2)}$-graph is a $\left(1 + 2\cdot\sin\left(\frac{\theta}{2}\right)\right)$-spanner of K_n.*

The upper bounds given in Theorem 3 and Corollary 4 are tight, as shown in Figure 5: we place a vertex v arbitrarily close to the upper corner of T_{uw} that is furthest from w. Likewise, we place a vertex v' arbitrarily close to the lower corner of T_{wu} that is furthest from u. Both shortest paths between u and w visit either v or v', so the path length is arbitrarily close to $(((1+\sin(\theta/2))/\cos(\theta/2)) \cdot \cos\alpha + \sin\alpha) \cdot |uw|$, showing that the upper bounds are tight.

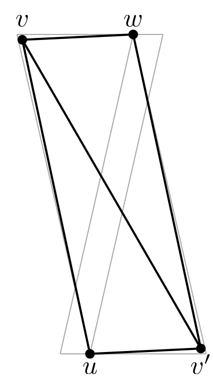

Figure 5: The lower bound for the $\theta_{(4k+2)}$-graph

4 Spanning Ratio of the $\theta_{(4k+4)}$-Graph

The $\theta_{(4k+2)}$-graph has the nice property that any line perpendicular to the bisector of a cone is parallel to the boundary of a cone (Lemma 1). As a result of this, if u, v, and w are vertices with v in one of the upper corners of T_{uw}, then T_{wv} is completely contained in T_{uw}. The $\theta_{(4k+4)}$-graph does not have this property. In this section, we show how to exploit this to construct a lower bound for the $\theta_{(4k+4)}$-graph whose spanning ratio exceeds the worst case spanning ratio of the $\theta_{(4k+2)}$-graph.

Theorem 5 *The worst case spanning ratio of the $\theta_{(4k+4)}$-graph is at least $1 + 2\tan\left(\frac{\theta}{2}\right) + 2\tan^2\left(\frac{\theta}{2}\right)$.*

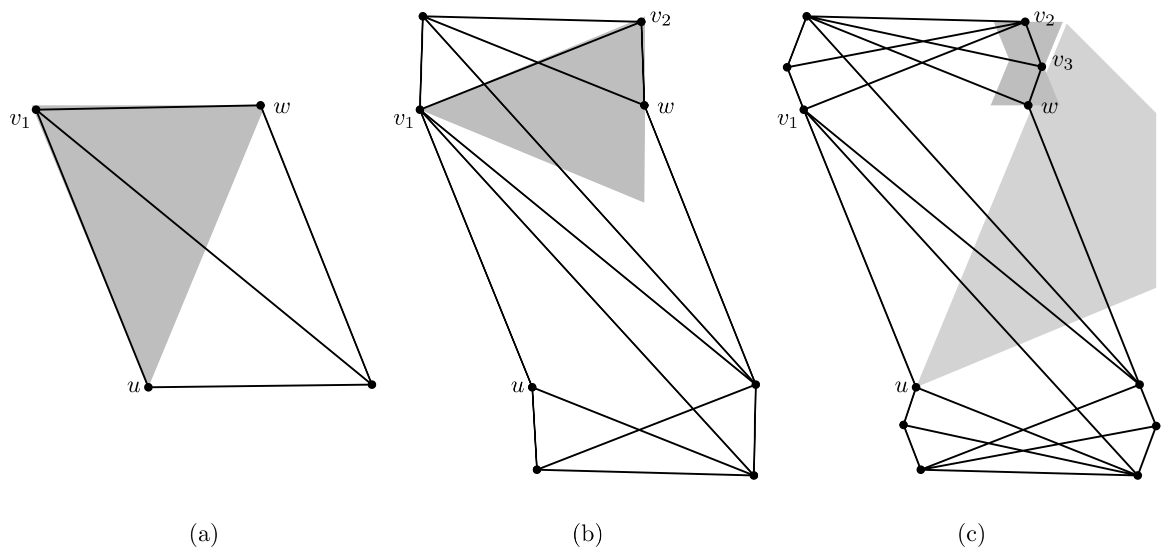

Figure 6: The construction of the lower bound for the $\theta_{(4k+4)}$-graph

Proof. We construct the lower bound example by extending the shortest path between two vertices u and w in three steps. We describe only how to extend one of the shortest paths between these vertices. To extend all shortest paths, the same modification is performed in each of the analogous cases, as shown in Figure 6.

First, we ensure that there is no edge between u and w by placing a vertex v_1 in the upper corner of T_{uw} that is furthest from w. See Figure 6a. Next, we place a vertex v_2 in the corner of $T_{v_1 w}$ that lies in the same cone of u as w and v_1. See Figure 6b. Finally, we place a vertex v_3 in the intersection of $T_{v_2 w}$ and $T_{w v_2}$ to ensure that there is no edge between v_2 and w. See Figure 6c. Note that we cannot place v_3 in the lower right corner of $T_{v_2 w}$ since this would cause an edge between u and v_3 to be added, creating a shortcut to w.

One of the shortest paths in the resulting graph visits u, v_1, v_2, v_3, and w. Thus, to obtain a lower bound for the $\theta_{(4k+4)}$-graph, we compute the length of this path.

Let m be the midpoint of the side of T_{uw} opposite u. By construction, we have that $\angle v_1 u m = \angle w u m = \angle v_2 v_1 w = \angle v_3 v_2 w = \angle v_3 w v_2 = \theta/2$. See Figure 7. We can express the various line segments as follows:

$$
\begin{aligned}
|uv_1| &= |uw| \\
|v_1 w| &= 2\sin\left(\frac{\theta}{2}\right)\cdot|uw| \\
|v_1 v_2| &= 2\tan\left(\frac{\theta}{2}\right)\cdot|uw| \\
|v_2 w| &= 2\sin\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)\cdot|uw| \\
|v_2 v_3| &= |v_3 w| = \tan^2\left(\frac{\theta}{2}\right)\cdot|uw|
\end{aligned}
$$

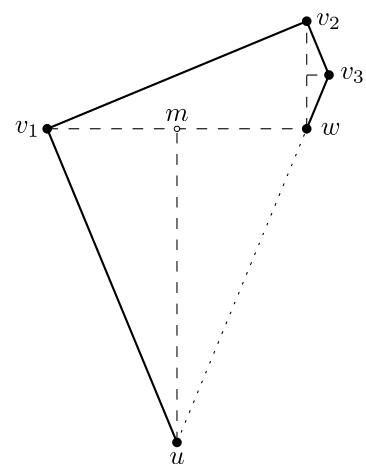

Figure 7: The lower bound for the $\theta_{(4k+4)}$-graph

Hence, the total length of the shortest path is $|uv_1| + |v_1 v_2| + |v_2 v_3| + |v_3 w| = (1 + 2\tan(\theta/2) + 2\tan^2(\theta/2)) \cdot |uw|$. □

Finally, we show that increasing the number of cones of a θ-graph by 2 from $4k+2$ to $4k+4$ increases the worst case spanning ratio.

Theorem 6 *The worst case spanning ratio of the $\theta_{(4k+4)}$-graph is greater than that of the $\theta_{(4k+2)}$-graph, for any integer $k \geq 1$.*

Proof. Recall that the worst case spanning ratio of the $\theta_{(4k+2)}$-graph is $1 + 2\sin(\pi/(4k+2))$ and that of the $\theta_{(4k+4)}$-graph is at least $1 + 2\tan(\pi/(4k+4)) + 2\tan^2(\pi/(4k+4))$. To prove the theorem, it suffices to show that $\tan(\pi/(4k+4)) + \tan^2(\pi/(4k+4))$ is greater than $\sin(\pi/(4k+2))$, for any integer $k \geq 1$.

For $x \in (0, \pi/6]$, it holds that $\sin x < x$, $\tan x > x$, and $\tan^2 x > x^2$. Since $k \geq 1$, both $\pi/(4k+2)$ and $\pi/(4k+4)$ are in the range $(0, \pi/6]$. Therefore, we have that:

$$\begin{aligned}\sin\left(\frac{\pi}{4k+2}\right) &< \frac{\pi}{4k+2} \\ &< \frac{\pi}{4k+4} + \left(\frac{\pi}{4k+4}\right)^2 \\ &< \tan\left(\frac{\pi}{4k+4}\right) + \tan^2\left(\frac{\pi}{4k+4}\right),\end{aligned}$$

as required. □

5 Conclusion

We showed that the $\theta_{(4k+2)}$-graph has a tight spanning ratio of $1 + 2\sin(\theta/2)$. This is the first time tight spanning ratios have been found for a large family of θ-graphs. Previously, the only θ-graph for which tight bounds were known was the θ_6-graph.

Furthermore, we showed that the $\theta_{(4k+4)}$-graph has a spanning ratio of at least $1 + 2\tan(\theta/2) + 2\tan^2(\theta/2)$. This result is somewhat surprising since, for equal values of k, the worst case spanning ratio of the $\theta_{(4k+4)}$-graph is greater than that of the $\theta_{(4k+2)}$-graph, showing that increasing the number of cones can make the spanning ratio worse.

There remain a number of open problems, such as finding lower bounds for the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph, and finding tight spanning ratios of the $\theta_{(4k+3)}$, $\theta_{(4k+4)}$, and $\theta_{(4k+5)}$-graphs. The best known upper bound for these graphs is $1/(1-2\sin(\theta/2))$. Furthermore, for the θ_4 and θ_5-graphs, neither upper nor lower bounds are known.

References

[1] N. Bonichon, C. Gavoille, N. Hanusse, and D. Ilcinkas. Connections between theta-graphs, Delaunay triangulations, and orthogonal surfaces. In *Proceedings of the 36th International Conference on Graph Theoretic Concepts in Computer Science (WG 2010)*, pages 266–278, 2010.

[2] P. Bose, R. Fagerberg, A. van Renssen, and S. Verdonschot. Competitive routing in the half-θ_6-graph. In *Proceedings of the 23rd ACM-SIAM Symposium on Discrete Algorithms (SODA 2012)*, pages 1319–1328, 2012.

[3] P. Chew. There are planar graphs almost as good as the complete graph. *Journal of Computer and System Sciences*, 39(2):205–219, 1989.

[4] K. Clarkson. Approximation algorithms for shortest path motion planning. In *Proceedings of the 19th Annual ACM Symposium on Theory of Computing (STOC 1987)*, pages 56–65, 1987.

[5] J. Keil. Approximating the complete Euclidean graph. In *Proceedings of the 1st Scandinavian Workshop on Algorithm Theory (SWAT 1988)*, pages 208–213, 1988.

[6] J. Ruppert and R. Seidel. Approximating the d-dimensional complete Euclidean graph. In *Proceedings of the 3rd Canadian Conference on Computational Geometry (CCCG 1991)*, pages 207–210, 1991.