Computing All Large Sums-of-Pairs in $\mathbb{R}^n$ and the Discrete Planar Two-Watchtower Problem*

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Abstract

We observe that Matoušek's algorithm for computing all dominances for a set $\mathcal{P}$ of n points in $\mathbb{R}^n$ can be employed for computing all pairs of points in such a set whose sum is greater or equal to a given point $\mathbf{a} \in \mathbb{R}^n$. We apply this observation to the decision problem of the discrete planar two-watchtower problem and obtain an improved solution.

Keywords: Algorithms, computational geometry, facility location.

1 Introduction

Let X be an $n \times n$ matrix of real numbers, and denote the i-th row of X by $x_{i,*}$. In [3] Matoušek presents an algorithm for computing an $n \times n$ matrix C, such that $c_{i,j} = n$ if and only if the i'th row of X is *dominated* by the j'th row of X, that is, $x_{i,k} \leq x_{j,k}$, for $k = 1, \ldots, n$. ($c_{i,j}$ is actually the number of coordinates k for which $x_{i,k} \leq x_{j,k}$.) The running time of Matoušek's algorithm is $O(n^{\frac{3}{2}} M(n)^{\frac{1}{2}})$, where $M(n)$ is the time required for multiplying two $n \times n$ matrices. Currently $M(n) = O(n^{2.376})$ (see [2]), and therefore the running time of his algorithm is $O(n^{2.688})$.

In this short paper we observe that Matoušek's algorithm can be employed to compute all pairs of rows in X, whose sum is greater or equal to a given n vector $\mathbf{a} = (a_1, \ldots, a_n)$, that is, all pairs of rows $x_{i,*}, x_{j,*}$, such that $x_{i,k} + x_{j,k} \geq a_k$, for $k = 1, \ldots, n$. The computation time remains $O(n^{\frac{3}{2}} M(n)^{\frac{1}{2}})$ (which is currently $O(n^{2.688})$).

Next, we apply this observation to the discrete planar two-watchtower problem, to obtain an improved solution to the corresponding decision problem. The input to the discrete version of the planar two-watchtower problem

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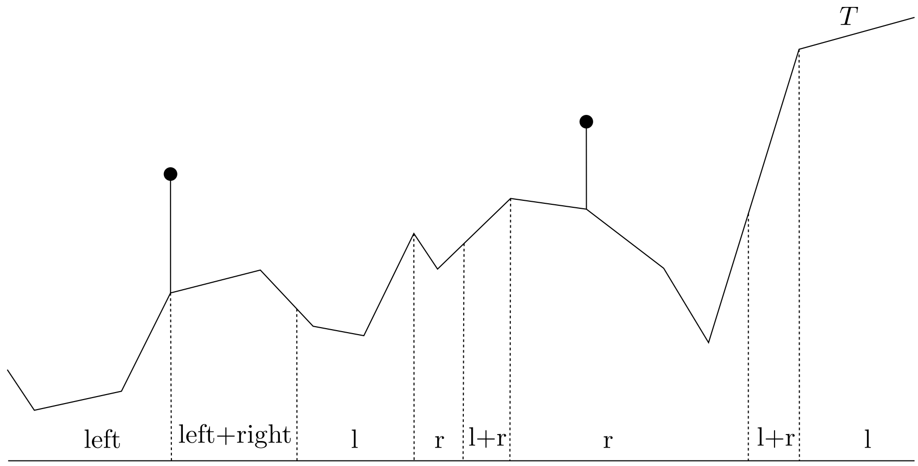

Figure 1: The two watchtower problem.

is an x-monotone polygonal line T, i.e., a 2-dimensional (polygonal) terrain. The goal is to place two watchtowers at two of the vertices of T, such that (i) each point of the terrain is seen from at least one of the watchtowers, and (ii) the height of the higher watchtower is as small as possible; see Figure 1.

The corresponding decision problem is thus, given a height h, determine whether the entire terrain can be viewed from two watchtowers of height h that are located at two of T's vertices. Bespamyatnikh et al. [1] present an $O(n^3)$ solution to this decision problem, where n is the number of vertices in T. We present an improved solution to this problem that is based on the algorithm for the large sums problem above. The running time of our solution is $O(n^{\frac{3}{2}}M(n)^{\frac{1}{2}})$ (which is currently $O(n^{2.688})$).

2 Computing all large sums-of-pairs in $\mathbb{R}^n$

Let X be an $n \times n$ matrix of real numbers, and let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n vector. We shall compute a matrix C, such that $c_{i,j} = n$ if and only if the sum of the i'th and j'th rows of X is at least $\mathbf{a}$, that is, $x_{i,k} + x_{j,k} \geq a_k$, for $k = 1, \ldots, n$. ($c_{i,j}$ is actually the number of coordinates k for which $x_{i,k} + x_{j,k} \geq a_k$.)

Let Y be the matrix obtained from X and $\mathbf{a}$ as follows. The i'th row of Y is $\mathbf{a} - x_{i,*}$. Notice that $x_{i,*} + x_{j,*}$ is greater or equal to $\mathbf{a}$ if and only if $y_{j,*}$ is dominated by $x_{i,*}$ and $y_{i,*}$ is dominated by $x_{j,*}$.

We form four $n \times n$ matrices as follows:

1. M^1 contains the first $n/2$ rows of X and the first $n/2$ rows of Y.

2. M^2 contains the first $n/2$ rows of X and the last $n/2$ rows of Y.

3. M^3 contains the last $n/2$ rows of X and the first $n/2$ rows of Y.

4. M^4 contains the last $n/2$ rows of X and the last $n/2$ rows of Y.

We now apply Matoušek's algorithm to each of the four matrices $M^1, \ldots, M^4$, obtaining matrices $C^1, \ldots, C^4$. The matrix C can now be easily computed from the matrices $C^1, \ldots, C^4$ as follows. Consider, e.g., the matrix C^3 (obtained from the matrix M^3). Then $c^3_{i,j} = n$, for $n/2 + 1 \leq i \leq n$ and $1 \leq j \leq n/2$, if and only if $y_{i-n/2,*}$ is dominated by $x_{j+n/2,*}$, or, in other words, $x_{i-n/2,*} + x_{j+n/2,*}$ is greater or equal to $\mathbf{a}$. Thus, the bottom left quadrant of C^3 is the top right quadrant of C. Similarly, the bottom left quadrant of C^1 is the top left quadrant of C, and the bottom left quadrants of C_2 and C_4 are the bottom left and bottom right quadrants of C, respectively.

Theorem 2.1 *Given a set of n points in $\mathbb{R}^n$ and a point $\mathbf{a} \in \mathbb{R}^n$, one can find all pairs of points whose sum, in each of the coordinates k, is at least a_k in $O(n^{3/2}M(n)^{1/2})$ time.*

Remark. Matoušek's algorithm can be adapted to solve the large sum-of-pairs problem directly.

3 The discrete planar two-watchtower problem

We solve the decision problem of the discrete planar two-watchtower problem. Given a 2-dimensional terrain $T = (v_0, \ldots, v_n)$ and a height h, determine whether the entire terrain can be viewed from two watchtowers of height h that are located at two of T's vertices.

For each vertex v_i of T, we shall place a watchtower of height h at v_i and compute the region R_i of T that is visible from this watchtower. We use the following easy and known observation (see Figure 2).

Observation: If $e_k = [v_{k-1}, v_k]$ is an edge of T lying to the right of v_i, then either (i) $e_k \cap R_i = \emptyset$ (e.g., in Figure 2, $e_8 \cap R_2 = \emptyset$), or (ii) $e_k \cap R_i = v_{k-1}$ (e.g., $e_{10} \cap R_2 = v_9$), or (iii) $e_k \cap R_i$ consists of a single line segment anchored at v_k (e.g., $e_9 \cap R_2$).

Thus, R_i is the union of n (possibly empty or degenerate) line segments, and R_i can be computed in $O(n)$ time.

Next we define an $n \times n$ matrix X. The i'th row of X is obtained from the region R_i as follows. Let $e_k = [v_{k-1}, v_k]$ be the k'th edge of T. Then

$$x_{i,k} = \frac{|e_k \cap R_i|}{|e_k|} .$$

Notice that if v_i and v_j are two vertices of T such that the entire terrain can be viewed from their watchtowers, then the sum of the i'th and j'th

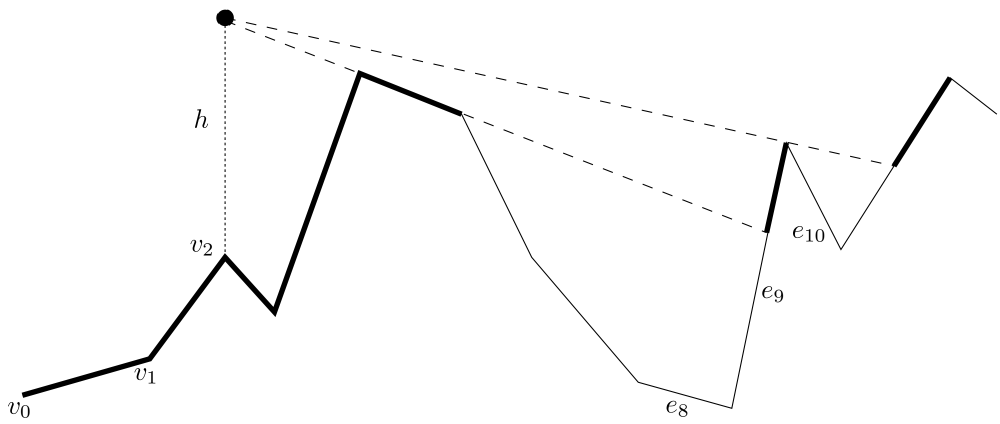

Figure 2: The region of T that is visible from the watchtower at v_2.

rows of X is greater or equal to $(1, \ldots, 1)$. That is, $x_{i,k} + x_{j,k} \geq 1$, for $k = 1, \ldots, n$.

In the following lemma we prove that the opposite statement is also true, that is

Lemma 3.1 *If the sum of the i'th and j'th rows of X is greater or equal to $(1, \ldots, 1)$, then the entire terrain can be viewed from the watchtowers at v_i and v_j.*

Proof: Assume $i < j$ and let $e_k = [v_{k-1}, v_k]$ be an edge of T. If e_k lies between v_i and v_j, then the part of e_k that is visible from the watchtower at v_i (resp. v_j) is a single segment anchored at v_k (resp. v_{k-1}) (see observation above). Therefore, since $x_{i,k} + x_{j,k} \geq 1$, the edge e_k is entirely covered by the watchtowers at v_i and v_j.

Assume now that e_k lies, e.g., to the right of v_j. We show that e_k is entirely covered by *one* of the two watchtowers. If $v_{k-1} = v_j$, then e_k is entirely covered by the watchtower at v_j, so assume $v_{k-1} \neq v_j$. Notice that by the observation above, if $0 < x_{i,k} < 1$ (resp. $0 < x_{j,k} < 1$), then the vertex v_{k-1} cannot be seen from the watchtower at v_i (resp. v_j), and, therefore, $x_{i,k-1}$ (resp. $x_{j,k-1}$) must be 0. Now, if e_k is not entirely covered by one of the two watchtowers, then, since we are assuming $x_{i,k} + x_{j,k} \geq 1$, we have $0 < x_{i,k} < 1$ and $0 < x_{j,k} < 1$, and therefore $x_{i,k-1} + x_{j,k-1} = 0$, contradicting our assumption. $\square$

According to the lemma above and to the paragraph preceding it, there exists a solution to our decision problem if and only if there exist two rows in X whose sum is greater or equal to $(1, \ldots, 1)$. In order to determine whether two such rows exist, we simply apply the algorithm from the preceding section with $\mathbf{a} = (1, \ldots, 1)$. Thus we obtain

Theorem 3.2 *The decision problem of the discrete planar two-watchtower problem can be solved in $O(n^{3/2}M(n)^{1/2})$ time.*

It is easy to verify that Megiddo's parametric search technique [4] can now be used to obtain an improved solution to the discrete planar two-watchtower problem. (The $\log^2 n$ factor in the bound below is the cost of applying parametric search.)

Theorem 3.3 *The discrete planar two-watchtower problem can be solved in $O(n^{3/2}M(n)^{1/2} \log^2 n)$ time.*

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