Proofs of Markov's Inequality & Chernoff's Bounds

Markov: For any non-negative r.v. X,

$$\Pr\{X \geq t \cdot E[X]\} \leq 1/t.$$

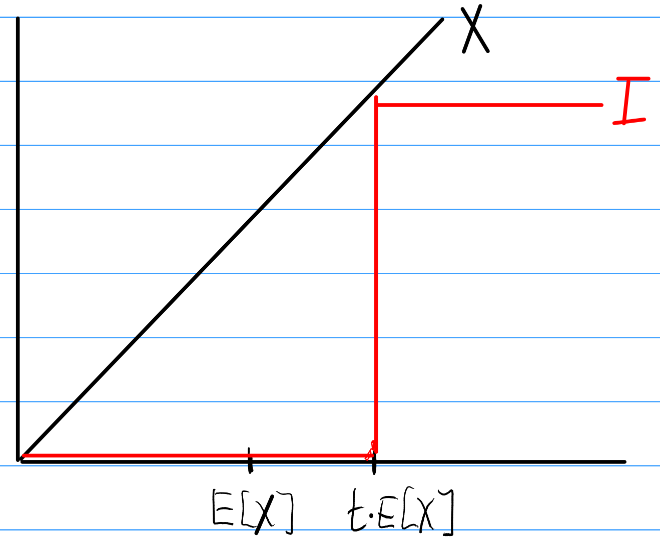
Proof: Let $I = \begin{cases} t \cdot E[X] & \text{if } X \geq t \cdot E[X] \\ 0 & \text{otherwise} \end{cases}$

Notice that $I \leq X$. (*)

$$E[I] = \Pr\{X \geq t \cdot E[X]\} \cdot t E[X] \underbrace{\leq \quad E[X]}_{\text{Because of } (*)} \quad (**)$$

Divide both sides of (**) by $t E[X]$ to get

$$\Pr\{X \geq t \cdot E[X]\} \leq 1/t.$$ QED



Recall that events A and B are independent if and only if $\Pr\{A \cap B\} = \Pr\{A\} \cdot \Pr\{B\}$.

Lemma: If X and Y are independent random variables, then

$$E[X \cdot Y] = E[X] \cdot E[Y].$$

Proof:

$$E[X \cdot Y] = \sum_x \sum_y \Pr\{X = x \text{ and } Y = y\} \cdot xy$$

$$= \sum_x \sum_y \Pr\{X = x\} \cdot \Pr\{Y = y\} \cdot xy$$ [because X and Y are indep.]

$$= \sum_x \Pr\{X = x\} \cdot x \cdot \left(\sum_y \Pr\{Y = y\} \cdot y \right)$$

$$= \sum_x \Pr\{X = x\} \cdot x \cdot E[Y]$$

$$= E[Y] \cdot \sum_x \Pr\{X = x\}$$

$$= E[Y] \cdot E[X]$$

QED.

Chernoff's Bounding Technique.

Idea: Markov's Inequality is too weak.

Instead of studying B, study $e^{c \cdot B}$.

If B is a little bit bigger than $E[B]$, the e^{cB} is a lot bigger that $E[e^{cB}]$.

Let $B = X_1 + X_2 + \cdots + X_n$ where the X_i's are independent Bernoulli(p) random variables.

$$E[e^{cX_i}] = p \cdot e^{c} + (1-p) \cdot e^{0} = p \cdot e^{c} + 1 - p = 1 + p(e^{c} - 1)$$

$$E[e^{cB}] = E[e^{cX_1} \cdot e^{cX_2} \cdots e^{cX_n}]$$

$$= E[e^{cX_1}] \cdot \cdots \cdot E[e^{cX_n}] = (1 + p(e^{c} - 1))^{n}$$

Next, we will use the Inequality:

$$1 + x \leq e^{x} \quad \text{for all } x,$$

so $E[e^{cB}] = (1 + p(e^{c} - 1))^{n} \leq e^{p(e^{c}-1)n}$

Remember Markov: $\Pr\{X \geq t \cdot E[X]\} \leq 1/t$.

Same as $\Pr\{X \geq q\} \leq \frac{E[X]}{q}$.

If $B \geq (1+\varepsilon)E[B]$, then $e^{cB} \geq e^{c(1+\varepsilon)np}$

So, $\Pr\{B \geq (1+\varepsilon)E[B]\}$

$$= \Pr\{\exp(cB) \geq \exp(c(1+\varepsilon)np)\}$$

$$\leq \frac{E[\exp(cB)]}{\exp(c(1+\varepsilon)np)}$$

$$\leq \frac{\exp(p(e^c-1)n)}{\exp(c(1+\varepsilon)np)}$$

$$= \frac{1}{\exp((c(1+\varepsilon) - e^c + 1)pn)}$$

Now we can choose c.

Let's take $c = \varepsilon^2/(1+\varepsilon)$

(**) $$= \frac{1}{\exp((\varepsilon^2 \underbrace{- e^{\varepsilon^2/(1+\varepsilon)} + 1}_{\geq 0 \text{ for all } 0 \leq \varepsilon \leq 1})pn)}$$

$$\leq \frac{1}{\exp(\varepsilon^2 pn)}, \quad \text{for any } 0 \leq \varepsilon \leq 1.$$

So, we get:

$$\Pr\{B \geq (1+\varepsilon)np\} \leq 1/e^{\varepsilon^2 np}, \text{ for any } 0<\varepsilon<1.$$

By choosing different values of c, we can get different bounds.

Taking $c = \ln(1+\varepsilon)$ gives the most common version:

$$\Pr\{B \geq (1+\varepsilon)np\} \leq \left(\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right)^{np}$$

Search for "Chernoff Bound" in Wikipedia for other versions.