COMP2804 Midterm Exam, Winter 2022

First Last 100000000

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This is a closed book exam. You are to do this exam on your own without consulting anyone else or using the internet.

Submit your answers at this URL: https://forms.gle/1mDZw4etNvKst8CK9
(Copy and paste the link into your browser if necessary.)

Marking Scheme: Each of the 17 questions is worth 1 mark.

Reminders:

- Binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Newton's Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

- Fibonacci numbers:

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2 \end{cases}$$

Question 1 (b)**.** The School of Computer Science has f full professors, a associate professors, t assistant professors, i instructors, and s students. The SCS Executive Committee must consist of

- 2 full (f) professors;
- 1 or 2 associate (a) professor;
- 1 assistant (t) professor;
- 1 instructor (i); and
- 2 students (s).

How many ways are there to form an SCS Executive Committee?

(1) $\binom{f}{2} \cdot \binom{a}{2} \cdot t \cdot i \cdot \binom{s}{2}$
(2) $\binom{f}{2} \cdot a \cdot t \cdot i \cdot \binom{s}{2}$
(3) $\binom{f}{2} \cdot a^2 \cdot t \cdot i \cdot \binom{s}{2}$
(4) $\binom{f}{2} \cdot 2a \cdot t \cdot i \cdot \binom{s}{2}$
(5) $\binom{f}{2} \cdot a \cdot t \cdot i \cdot \binom{s}{2} + \binom{f}{2} \cdot \binom{a}{2} \cdot t \cdot i \cdot \binom{s}{2}$

Question 2 (b)**.** A class contains $n \geq 2$ distinct students and wants to send a group of $k \leq n-2$ students on a field trip. Two of these students, Fred and Steve, are bullies. The rest of the students are not bullies.

How many ways are there to choose a set of k students that does not contain two bullies?

(1) $\binom{n-1}{k} + \binom{n-2}{k}$
(2) $\binom{n-2}{k}$
(3) $\binom{n-2}{k-1} \cdot (n-k-1)$
(4) $\binom{n-1}{k}$
(5) $\binom{n-2}{k} + 2 \cdot \binom{n-2}{k-1}$

Question 3 (b)**.** Piper (P) and Marley (M) are getting married. In addition to Piper and Marley, the wedding party has p of Piper's friends $P_1, \ldots, P_p$ and m of Marley's friends $M_1, \ldots, M_m$. Piper and Marley have no common friends, so $\{P_1, \ldots, P_p\} \cap \{M_1, \ldots, M_m\} = \emptyset$.

It's later in the evening, everyone has been drinking, and Piper and Marley had a fight. It's time to take a wedding photo and we want to line up the entire wedding party without having Piper and Marley standing next to each other. For example we could line them up like $M, P_1, \ldots, P_p, P, M_1, \ldots, M_m$.

How many ways are there to line up to the group so that Piper and Marley are *not* standing beside each other?

(1) $p!m!$
(2) $2p!m!$
(3) $4p!m!$
(4) $p!m! + p!m!$
(5) $(p+m+1)!(p+m)$

Question 4 (b)**.** Let's go to the animal shelter to take home some cats. At the shelter there are $b \geq 5$ black cats $B_1, \ldots, B_b$ and $w \geq 5$ white cats $W_1, \ldots, W_w$. All the cats are distinct; we can distinguish between any two cats, even if they have the same colour.

How many ways are there to take home 5 cats so that we take home an odd number of white cats?

(1) $\binom{b+w}{5}$
(2) $\binom{b}{5} + \binom{b}{3} \cdot \binom{w}{2} + \binom{b}{1} \cdot \binom{w}{4}$
(3) $\binom{w}{5} + \binom{w}{3} \cdot \binom{b}{2} + \binom{w}{1} \cdot \binom{b}{4}$
(4) $\binom{w}{5} + \binom{w}{4} \cdot b + \binom{w}{3} \cdot \binom{b}{2}$
(5) $\sum_{k=0}^{5} \binom{w}{k} \cdot \binom{b}{5-k}$

Question 5 (b)**.** Let $n \geq 5$. How many strings of length n over the alphabet $\{a, b, c\}$ begin with abc or end with bb?

(1) 3^n
(2) 3^{n-5}
(3) $3^{n-2} - 3^{n-5}$
(4) $3^{n-3} + 3^{n-2} - 3^{n-5}$
(5) $3^n - 3^{n/2}$

Question 6 (b)**.** How many strings can be obtained by rearranging the letters of the word SIMSANTEATERS

(1) $13!$
(2) $13!/48$
(3) $13 \cdot 12 \cdot \binom{11}{3} \cdot \binom{9}{2} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot 2 \cdot 1$
(4) $13 \cdot 12 \cdot \binom{11}{2} \cdot \binom{9}{3} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot 2 \cdot 1$
(5) $13 \cdot 12 \cdot \binom{11}{2} \cdot \binom{9}{2} \cdot \binom{7}{3} \cdot \binom{5}{3} \cdot 2 \cdot 1$

Question 7 (b)**.** What does $\sum_{k=1}^{n} \binom{n}{k} 2^{n-k}$ count?

(1) The number of strings of length n over the alphabet $\{a, b\}$
(2) The number of strings of length n over the alphabet $\{a, b\}$ that contain at least one a
(3) The number of strings of length n over the alphabet $\{a, b, c\}$
(4) The number of strings of length n over the alphabet $\{a, b, c\}$ that contain no a
(5) The number of strings of length n over the alphabet $\{a, b, c\}$ that contain at least one a

Question 8 (b)**.** I have a jar with 55 balls numbered $1, \ldots, 55$. I want to take balls out of the jar until I find two different pairs of balls $\{b_1, b_2\}$ and $\{b_3, b_4\}$ such that $b_1 - b_2 = b_3 - b_4$.

The fewest balls I must take out before I am guaranteed this will happen is:

(1) 4 balls
(2) 11 balls
(3) 15 balls
(4) 27 balls
(5) 55 balls

Question 9 (b)**.** What is the coefficient of $x^{13}y^9$ in the expansion of $(3x - 2y)^{22}$?

(1) $\binom{22}{9} \cdot 3^{13} \cdot 2^9$
(2) $-\binom{22}{11} \cdot 3^{13} \cdot 2^9$
(3) $\binom{22}{12} \cdot 3^{13} \cdot 2^9$
(4) $-\binom{22}{13} \cdot 3^{13} \cdot 2^9$
(5) None of the other answers is correct

Question 10 (b)**.** A string over the alphabet $\{a, b, c\}$ is called *great* if it does not contain bc or ba. Let $n \geq 2$. How many great strings of length n are there?

(1) 2^n
(2) $2^{n+1} - 1$
(3) 3^n
(4) $3^n - 2^n$
(5) f_{n+1}

Question 11 (b)**.** A bitstring is called 00-free if it does not contain two 0s next to each other. In class we have seen that, for any $m \geq 1$, the number of 00-free bitstrings of length m is equal to the $(m + 2)$th Fibonacci number f_{m+2}.

What is the number of 00-free bitstrings of length 30 that have 1 at position 9? (The positions are numbered $1, 2, \ldots, 30$.)

(1) $f_7 \cdot f_{20}$
(2) $f_8 \cdot f_{21}$
(3) $f_9 \cdot f_{22}$
(4) $f_{10} \cdot f_{23}$
(5) None of the other answers is correct

Question 12 (b)**.** A string over the alphabet $\{x, y, z\}$ is called *fabulous* if it does not contain xyz, xyx, or xx. For $n \geq 1$, let A_n denote the number of fabulous strings of length n. Which of the following is true for any $n \geq 4$?

(1) $A_n = A_{n-1} + A_{n-2} + A_{n-3}$
(2) $A_n = 2A_{n-1} + A_{n-2} + A_{n-3}$
(3) $A_n = 2A_{n-1} + 2A_{n-2} + A_{n-3}$
(4) $A_n = 2A_{n-1} + 2A_{n-2} + 2A_{n-3}$
(5) None of the other answers is correct

Question 13 (b)**.** Consider the following recursive function $\text{BAR}(n)$:

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BAR(n) :
  if n ≤ 3 then
    return n
  return BAR(n − 1) + BAR(n − 3)
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When running $\text{BAR}(46)$ how many calls are there to $\text{FU}(41)$?

(1) 4
(2) 6
(3) 8
(4) 9
(5) 10

Question 14 (b)**.** You are given an infinite supply of red marbles and an infinite supply of blue marbles. Let S_n be the number of ways of placing n of these marbles in a line so that you never have three red marbles in a row. Which of the following is true, for any $n \geq 3$?

(1) $S_n = 2S_{n-1}$
(2) $S_n = 4S_{n-3}$
(3) $S_n = S_{n-1} + S_{n-2} + S_{n-3}$
(4) $S_n = S_{n-1} + 2S_{n-2} + S_{n-3}$
(5) $S_n = 2^n$

Question 15 (b)**.** You toss a fair coin 12 times. Define the event:

$$A = \text{"the results of the last three flips are equal"}$$

What is $\Pr(A)$?

(1) 1/2
(2) 1/4
(3) 1/6
(4) 1/8
(5) 1/16

Question 16 (b)**.** A bag contains r red balls, b blue balls, and g green balls. We reach into the bag and choose a uniformly random subset of 2 balls. Define the event:

$$A = \text{“this subset has no blue balls”}$$

What is $\Pr(A)$?

(1) $1/\binom{r+g+b}{2}$
(2) $rg/\binom{r+g+b}{2}$
(3) $\binom{r+g}{2}/\binom{r+g+b}{2}$
(4) $(r+g)^2/(r+g+b)^2$
(5) rg/rgb

Question 17 (b)**.** A bag contains r red balls, b blue balls, and g green balls. We reach into the bag and choose a uniformly random subset of 2 balls. Define the event:

$$A = \text{“the subset contains one red ball and one green ball”}$$

What is $\Pr(A)$?

(1) $1/\binom{r+g+b}{2}$
(2) $rg/\binom{r+g+b}{2}$
(3) $\binom{r+g}{2}/\binom{r+g+b}{2}$
(4) $(r+g)^2/(r+g+b)^2$
(5) rg/rgb