

# Carleton University

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Final  
Examination  
Fall 2015

DURATION: 2 HOURS

No. of students: 195

Department Name & Course Number: **Computer Science COMP 2804A**

Course Instructor: Michiel Smid

Authorized memoranda:  
Calculator

**Students MUST count the number of pages in this examination question paper before beginning to write, and report any discrepancy to the proctor. This question paper has 12 pages (not including the cover page).**

**This examination question paper MAY be taken from the examination room.**

**In addition to this question paper, students require:**

an examination booklet: no  
a Scantron sheet: yes

**Instructions:**

1. All questions must be answered on the scantron sheet.
2. Nick is not allowed to eat bananas during the exam.

**Marking scheme:** Each of the 25 questions is worth 1 mark.

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ .
- Newton:  $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$ .
- For  $0 < x < 1$ ,  $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$ .
- Geometric distribution: Assume an experiment has a success probability of  $p$ . We perform the experiment until it is successful for the first time. The expected number of times we perform the experiment is  $1/p$ .

1. Consider a set  $S$  consisting of 20 integers; 5 of these are strictly positive and the other 15 integers in  $S$  are strictly negative. What is the number of 3-element subsets of  $S$  having the property that the product of the 3 elements in the subset is negative?

- (a)  $\binom{20}{3}$
- (b)  $\binom{15}{3}$
- (c)  $\binom{15}{3} + 15 \cdot \binom{5}{2}$
- (d)  $\binom{15}{3} + \binom{15}{2} \cdot 5 + 15 \cdot \binom{5}{2}$

2. Consider a set  $S$  consisting of 20 integers. The integer 0 is an element of  $S$ , 9 of the elements in  $S$  are strictly positive, and the remaining 10 elements are strictly negative. What is the number of 7-element subsets of  $S$  having the property that the product of the 7 elements in the subset is equal to 0?

- (a)  $\binom{19}{6}$
- (b)  $\binom{19}{7}$
- (c)  $\binom{20}{6}$
- (d)  $\binom{20}{7}$

3. How many bitstrings  $s_1s_2 \cdots s_{20}$  of length 20 have the property that  $s_1s_2s_3 = 000$  or  $s_2s_3s_4 = 000$ ?

- (a)  $2^{17} - 2^{15}$
- (b)  $2^{17} - 2^{16}$
- (c)  $2^{18} - 2^{16}$
- (d)  $2^{18} - 2^{17}$

4. What is the coefficient of  $x^{15}y^5$  in the expansion of

$$(-3x + 5y)^{20}.$$

- (a)  $\binom{20}{5} \cdot 3^{15} \cdot 5^5$
- (b)  $-\binom{20}{5} \cdot 3^{15} \cdot 5^5$
- (c)  $\binom{20}{5} \cdot 5^{15} \cdot 3^5$
- (d)  $-\binom{20}{5} \cdot 5^{15} \cdot 3^5$

5. In a group of 30 people,

- 10 are blond,
- 20 have green eyes,
- 9 are not blond and do not have green eyes.

How many people in this group are blond and have green eyes?

- (a) 8
- (b) 9
- (c) 10
- (d) 11

6. Consider strings of characters, each character being  $a$ ,  $b$ , or  $c$ , that contain at least one  $a$ . Let  $S_n$  be the number of such strings having length  $n$ . Which of the following is true?

- (a)  $S_n = n \cdot 2^{n-1}$
- (b)  $S_n = n \cdot 3^{n-1}$
- (c)  $S_n = 3^n - n$
- (d)  $S_n = 3^n - 2^n$

7. Consider strings of characters, each character being  $a$ ,  $b$ , or  $c$ , that contain at least one  $a$ . Let  $S_n$  be the number of such strings having length  $n$ . Which of the following is true?

- (a)  $S_n = 2 \cdot S_{n-1} + 3^{n-1}$
- (b)  $S_n = 2 \cdot S_{n-1} + 2 \cdot S_{n-2}$
- (c)  $S_n = 3 \cdot S_{n-1}$
- (d) None of the above.

8. Consider the following recursive function:

$$\begin{aligned} f(0) &= 7, \\ f(n) &= 2 \cdot f(n-1) + 1 \text{ for all integers } n \geq 1. \end{aligned}$$

Which of the following is true?

- (a)  $f(n) = 8n + 7$
- (b)  $f(n) = 4n^2 + 4n + 7$
- (c)  $f(n) = 2^{n+3} - 1$ .
- (d) None of the above.

9. Consider the recursive algorithm FIB, which takes as input an integer  $n \geq 0$ :

```
Algorithm FIB( $n$ ):  
if  $n = 0$  or  $n = 1$   
then  $f = n$   
else  $f = \text{FIB}(n - 1) + \text{FIB}(n - 2)$   
endif;  
return  $f$ 
```

If we run algorithm FIB(18), how many calls are there to FIB(14)?

- (a) 4
  - (b) 5
  - (c) 6
  - (d) 7
10. What is the number of bitstrings of length  $n$  that contain 00 or 11?
- (a)  $2 \cdot (n - 1) \cdot 2^{n-2}$
  - (b)  $2 \cdot n \cdot 2^{n-1}$
  - (c)  $2^n - 4$
  - (d)  $2^n - 2$
11. Nick gets 75 bananas for his birthday. He decides to eat them all over a period of 5 days. In order to do this, Nick makes a banana-schedule, which specifies the number of bananas he is going to eat on the first day, on the second day, etc., up to the fifth day. For example, (20, 20, 10, 20, 5), (40, 13, 0, 20, 2), and (40, 13, 20, 2, 0) are three different banana-schedules. What is the total number of banana-schedules?
- (a)  $\binom{79}{4}$
  - (b)  $\binom{79}{5}$
  - (c)  $\binom{80}{4}$
  - (d)  $\binom{80}{5}$

12. A bowl contains 5 red balls and 7 blue balls. We choose a uniformly random subset of 3 balls. Define the event

$$A = \text{“exactly 2 of the chosen balls are red”}.$$

What is  $\Pr(A)$ ?

- (a)  $\frac{\binom{5}{2}}{\binom{12}{3}}$
- (b)  $\frac{5 \cdot \binom{7}{2}}{\binom{12}{3}}$
- (c)  $\frac{\binom{5}{2} \cdot 7}{\binom{12}{3}}$
- (d)  $\frac{\binom{7}{2}}{\binom{12}{3}}$

13. Consider a uniformly random bitstring of length 5. Define the events

$$\begin{aligned} A &= \text{“the first three bits are 101 or 110”}, \\ B &= \text{“the last three bits are 111”}. \end{aligned}$$

Which of the following is true?

- (a) The events  $A$  and  $B$  are independent.
  - (b) The events  $A$  and  $B$  are not independent.
  - (c) None of the above.
14. Let  $n$  be the number of students who are writing this exam. Each of these students has a uniformly random birthday, which is independent of the birthdays of the other students. We ignore leap years; thus, the year has 365 days. Define the event

$$A = \text{“at least one student’s birthday is on December 21”}.$$

What is  $\Pr(A)$ ?

- (a)  $n \cdot (1/365) \cdot (364/365)^{n-1}$
- (b)  $365 \cdot n \cdot (364/365)^{n-1}$
- (c)  $1 - (1/365)^n$
- (d)  $1 - (364/365)^n$

Questions 15, 16, and 17 are based on the following:

- The course COMP 9999 runs over a period of one year, starting on January 1 and ending on December 31. There is one lecture every day; thus, the total number of lectures is 365.
- Dania and Nick take this course. Dania's birthday is on November 19. Nick's birthday is on December 3.
- Professor G. Ruesome teaches the course. Professor Ruesome decides to have 20 quizzes during the year. For this, he chooses a uniformly random subset of 20 days; the quizzes will be on the 20 chosen days. (It is possible that there is a quiz on January 1.)

15. Determine  $\Pr(A)$ , where  $A$  is the event

$A =$  "There is a quiz on Dania's birthday and there is a quiz on Nick's birthday".

- (a)  $1 - \binom{363}{20} / \binom{365}{20}$
- (b)  $\binom{363}{18} / \binom{365}{20}$
- (c)  $\binom{365}{18} / \binom{365}{20}$
- (d) None of the above.

16. Determine the conditional probability  $\Pr(B | C)$ , where  $B$  and  $C$  are the events

$B =$  "there is a quiz on Nick's birthday",  
 $C =$  "there are exactly 5 quizzes in December".

- (a)  $4/31$
- (b)  $5/31$
- (c)  $4/32$
- (d)  $5/32$

17. At the beginning of each of the 365 lectures, Nick flips a fair and independent coin twice. If the coin comes up heads twice, then Nick eats 3 bananas during the lecture; otherwise, Nick eats 5 bananas during the lecture. Let  $X$  be the total number of bananas that Nick eats during the 365 lectures of the course COMP 9999. What is the expected value  $\mathbb{E}(X)$  of  $X$ ?

- (a)  $\frac{9 \cdot 365}{2}$
- (b)  $\frac{7 \cdot 365}{2}$
- (c)  $\frac{5 \cdot 365}{2}$
- (d)  $4 \cdot 365$

Both Questions 18 and 19 are based on the following: The  $n$  students  $S_1, S_2, \dots, S_n$  decide to organize a Secret Santa: They take a uniformly random permutation  $P_1, P_2, \dots, P_n$  of  $S_1, S_2, \dots, S_n$ . For each  $i = 1, 2, \dots, n$ , student  $S_i$  buys a gift and gives it, anonymously, to student  $P_i$ .

18. Let  $X$  be the number of students who give their gift to themselves. What is the expected value  $\mathbb{E}(X)$  of the random variable  $X$ ?

*Hint:* Use an indicator random variable for each student.

- (a) 1
- (b) 2
- (c)  $1 + 1/n$
- (d)  $2 + 1/n$

19. For each  $i = 1, 2, \dots, n$ , let  $v_i$  be the value (in dollars) of the gift that student  $S_i$  buys. Let  $Y$  be the value of the gift that student  $S_1$  receives, and let  $Z$  be the value of the gift that student  $S_2$  receives. What is

$$\mathbb{E}(2 \cdot Y - Z)?$$

- (a)  $2v_1 - v_2$
- (b)  $\sum_{i=3}^n v_i/n$
- (c)  $\sum_{i=1}^n v_i/n$
- (d)  $2 \sum_{i=2}^n v_i/n - (v_1/n + \sum_{i=3}^n v_i/n)$



20. You repeatedly, and independently, flip three fair coins, until there are exactly two heads among the three flips. Define the random variable  $X$  to be the total number of coin flips. For example, if the coin flips result in

$$(TTT), (THT), (HTH), (HTH),$$

then  $X = 12$ .

What is the expected value  $\mathbb{E}(X)$  of  $X$ ?

- (a)  $3/8$
  - (b)  $8/3$
  - (c)  $8$
  - (d)  $12$
21. Let  $n$  be an integer with  $n \geq 3$ . Consider a bitstring of length  $n$ , in which each bit is 0 with probability  $1/3$  (and, thus, 1 with probability  $2/3$ ), independently of the other bits. Let  $X$  be the number of occurrences of 010 in this bitstring. For example, if the bitstring is

$$0010100100,$$

then  $X = 3$ .

What is the expected value  $\mathbb{E}(X)$  of  $X$ ?

*Hint:* Use indicator random variables.

- (a)  $2n/27$
- (b)  $2(n - 2)/27$
- (c)  $(n - 2)/8$
- (d)  $n/8$

22. Let  $S = \{1, 2, \dots, n\}$  and let  $T$  be a set of  $m$  unordered pairs of distinct elements of  $S$ . Thus,

$$T \subseteq \{\{i, j\} : 1 \leq i < j \leq n\}.$$

Consider a coin that comes up heads with probability  $1/3$  and, thus, tails with probability  $2/3$ . For each element of  $S$ , flip the coin, and let  $S'$  be the set consisting of all elements of  $S$  whose coin flip resulted in heads. Let  $T'$  be the set consisting of all elements  $\{i, j\}$  in  $T$  for which both  $i$  and  $j$  are in  $S'$ .

Let  $X$  be the size of the set  $T'$ . What is the expected value  $\mathbb{E}(X)$  of  $X$ ?

*Hint:* Use indicator random variables.

- (a)  $n/9$
- (b)  $m/9$
- (c)  $4n/9$
- (d)  $4m/9$

23. Let  $S$  be a uniformly random 2-element subset of  $\{1, 2, 3, 4\}$ , and let  $X$  be the number of elements of  $S$  that are even. What is the expected value  $\mathbb{E}(X)$  of  $X$ ?

- (a) 1
- (b)  $3/2$
- (c) 2
- (d)  $5/2$

24. Consider a coin that comes up heads with probability  $1/3$  and, thus, tails with probability  $2/3$ . Consider the following recursive algorithm HEADS, which takes as input a positive integer  $k$ :

```
Algorithm HEADS( $k$ ):  
  // all coin flips made are mutually independent  
  flip the coin;  
  if the coin came up heads  
  then return  $k + 1$   
  else HEADS( $k + 1$ )  
  endif
```

You run algorithm HEADS(1), i.e., with  $k = 1$ . Define the random variable  $X$  to be the value of the output of this algorithm. Let  $m \geq 1$  be an integer. What is  $\Pr(X = m + 1)$ ?

- (a)  $2^{m-1}/3^m$
  - (b)  $(2/3)^m$
  - (c)  $2^m/3^{m+1}$
  - (d)  $(2/3)^{m+1}$
25. Who discovered Newton's Binomial Theorem?
- (a) Professor Binomial.
  - (b) Justin Bieber.
  - (c) Isaac Newton.
  - (d) Professor G. Ruesome (the guy teaching COMP 9999).



