Sorting and Sorting Lower Bounds

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COMP2402/2002

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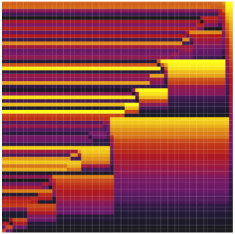

The Merge Sort Algorithm

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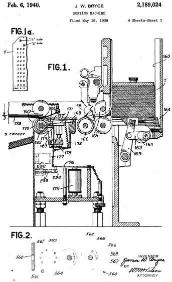

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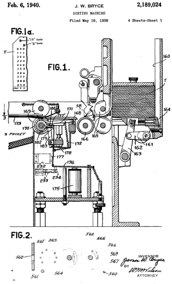

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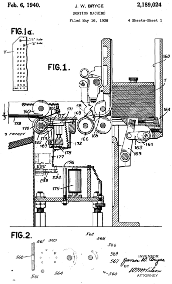

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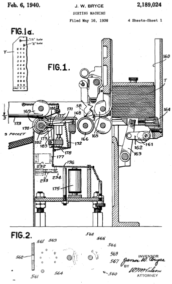

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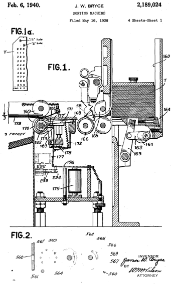

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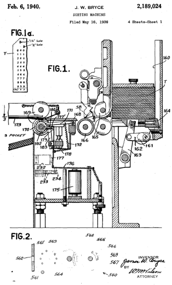

Mergesort

- To sort $\mathtt{a}[0], \ldots, \mathtt{a}[\mathtt{n}-1]$:

- $\langle 9, 3, 5, 2, 1, 8, 7, 0, 6, 4 \rangle$
- $\langle 9, 3, 5, 2, 1 \rangle \langle 8, 7, 0, 6, 4 \rangle$

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- To sort $\mathtt{a}[0],\ldots,\mathtt{a}[\mathtt{n}-1]$:
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- $\langle 9, 3, 5, 2, 1, 8, 7, 0, 6, 4 \rangle$
- $\langle 1, 2, 3, 5, 9 \rangle \langle 0, 4, 6, 7, 8 \rangle$
- $\langle 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \rangle$

Mergesort

```
<T> void mergeSort(T[] a, Comparator<T> c) {
  if (a.length <= 1) return;
  T[] a0 = Arrays.copyOfRange(a, 0, a.length/2);
  T[] a1 = Arrays.copyOfRange(a, a.length/2, a.length);
  mergeSort(a0, c);
  mergeSort(a1, c);
  merge(a0, a1, a, c);
}
```

Merging two sorted arrays

- To merge two sorted arrays (or lists) `a` and `b` we scan them sequentially

```
<T> void merge(T[] a0, T[] a1, T[] a, Comparator<T> c) {
  int i0 = 0, i1 = 0;
  for (int i = 0; i < a.length; i++) {
    if (i0 == a0.length)
      a[i] = a1[i1++];
    else if (i1 == a1.length)
      a[i] = a0[i0++];
    else if (compare(a0[i0], a1[i1]) < 0)
      a[i] = a0[i0++];
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```

- Takes $O(n)$ time

Analysis of Mergesort

- Mergesort $\mathtt{a}[0], \ldots, \mathtt{a}[\mathtt{n}-1]$:
- Let $T(\mathtt{n})$ be the time to run merge sort on an array of length n

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[1] Cheating a bit here, assuming n is a power of 2.

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- Mergesort $\mathtt{a}[0], \ldots, \mathtt{a}[\mathtt{n}-1]$:
  1. sort $\mathtt{a}[0], \ldots, \mathtt{a}[\mathtt{n}/2]$ (recursively)

- Let $T(\mathtt{n})$ be the time to run merge sort on an array of length n
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- Let $T(\mathtt{n})$ be the time to run merge sort on an array of length n
- Step 1 Takes $T(\mathtt{n}/2)$ time
- Step 2 Takes $T(\mathtt{n}/2)$ time
- Step 3 Takes $O(\mathtt{n})$ time
- $T(\mathtt{n}) = O(\mathtt{n}) + 2T(\mathtt{n}/2)$[1]

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The Mergesort recurrence

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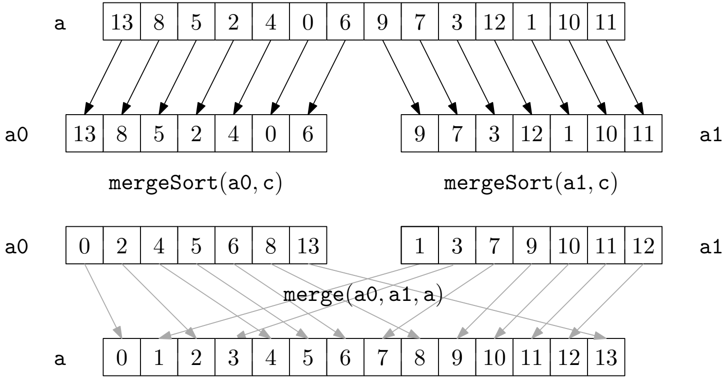
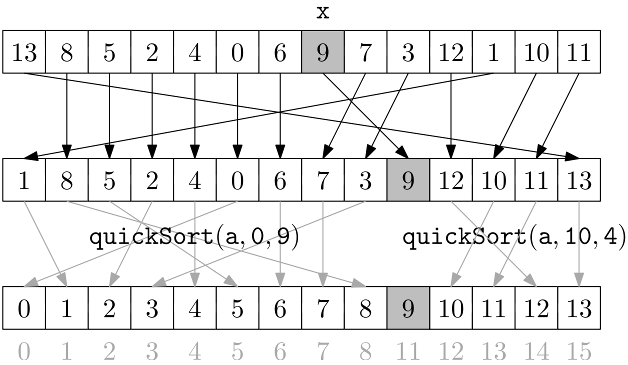
- $T(\mathtt{n}) = O(\mathtt{n}) + O(\mathtt{n}) + O(\mathtt{n}) + \cdots + O(\mathtt{n})$
- $T(\mathtt{n}) = O(\mathtt{n}\log\mathtt{n})$
- **Theorem:** The Mergesort algorithm can sort an array of $\mathtt{n}$ items in $O(\mathtt{n}\log\mathtt{n})$ time

Reminder: Mergesort



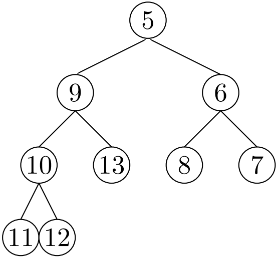
- Mergesort sorts an array of $\mathtt{n}$ elements in $O(\mathtt{n} \log \mathtt{n})$ worst-case time using at most $\mathtt{n} \log \mathtt{n}$ comparisons

Reminder: Quicksort



- Quicksort sorts an array of n elements in $O(n \log n)$ expected time using at most $1.38 n \log n$ expected comparisons

Reminder: Heapsort



5	9	6	10	13	8	7	11	12	4	3	2	1	0
0	1	2	3	4	5	6	7	8	11	12	13	14	15

- Heapsort sorts an array of $\mathtt{n}$ elements in $O(\mathtt{n}\log\mathtt{n})$ worst-case time using at most $2\mathtt{n}\log\mathtt{n}$ comparisons

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  - Answer: No and yes

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  - But this comes at a price
    - Every comparison-based sorting algorithm takes $\Omega(\mathtt{n}\log\mathtt{n})$ time for some input

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    - if $a[u.i] < a[u.j]$ then $\texttt{u} = \texttt{u.left}$ else $\texttt{u} = \texttt{u.right}$
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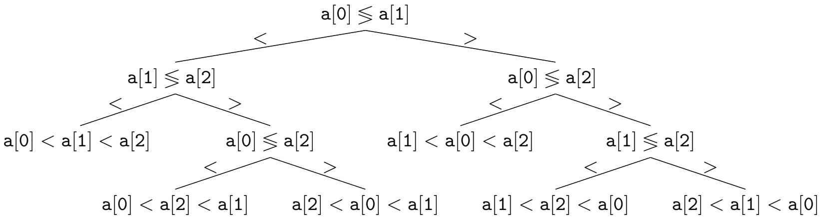
Comparison tree example



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$$\begin{aligned}
\log_2 \mathtt{n}! &= \log_2(\mathtt{n}) + \log_2(\mathtt{n}-1) + \cdots + \log_2(1) \\
&\geq \log_2(\mathtt{n}) + \cdots + \log_2(\mathtt{n}/2) \\
&\geq \log_2(\mathtt{n}/2) + \cdots + \log_2(\mathtt{n}/2) \\
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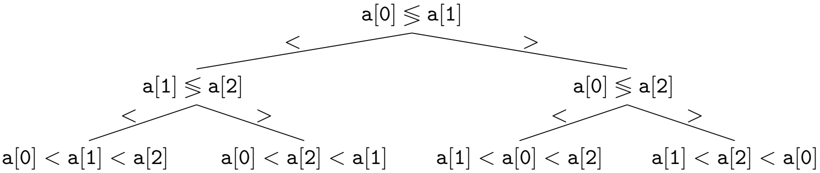
\end{aligned}$$

- Lower bound can be improved to $\mathtt{n}\log\mathtt{n} - O(\mathtt{n})$

Comparison tree lower bound



$a[0] \lessgtr a[1]$

$<$ $>$

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$<$ $>$ $<$ $>$

$a[0] < a[1] < a[2]$ $a[0] < a[2] < a[1]$ $a[1] < a[0] < a[2]$ $a[1] < a[2] < a[0]$

- Does not sort correctly because

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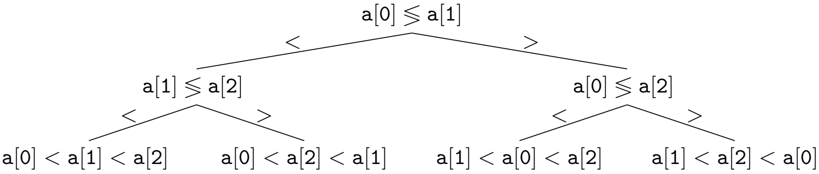

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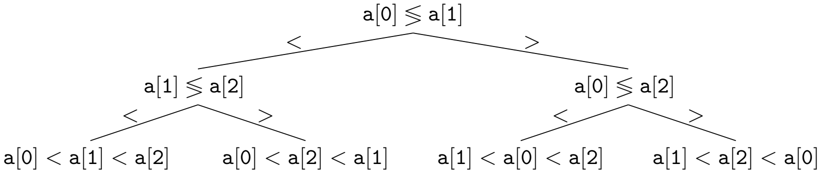

- Does not sort correctly because
  - $3! = 3 \cdot 2 \cdot 1 = 6$
  - this tree has only $4 < 6$ leaves

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- Every deterministic comparison-based sorting algorithm $\mathcal{A}$ that can sort every array of n elements defines a comparison tree $T_{\mathcal{A}}$ that sorts

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- Every deterministic comparison-based sorting algorithm $\mathcal{A}$ that can sort every array of n elements defines a comparison tree $T_{\mathcal{A}}$ that sorts
- The height of $T_{\mathcal{A}}$ is equal to the (worst-case) number of comparisons that $\mathcal{A}$ performs

Comparison-based sorting and comparison trees

- Every deterministic comparison-based sorting algorithm $\mathcal{A}$ that can sort every array of n elements defines a comparison tree $T_{\mathcal{A}}$ that sorts
- The height of $T_{\mathcal{A}}$ is equal to the (worst-case) number of comparisons that $\mathcal{A}$ performs
- **Theorem:** For every deterministic comparison-based sorting algorithm $\mathcal{A}$, there exists an input such that $\mathcal{A}$ requires $\Omega(\mathtt{n} \log \mathtt{n})$ comparisons
- **Theorem:** For every comparison-based sorting algorithm $\mathcal{A}$, the expectedd number of comparisons performed by $\mathcal{A}$ while sorting a random permutation is $\Omega(\mathtt{n} \log \mathtt{n})$

Summary

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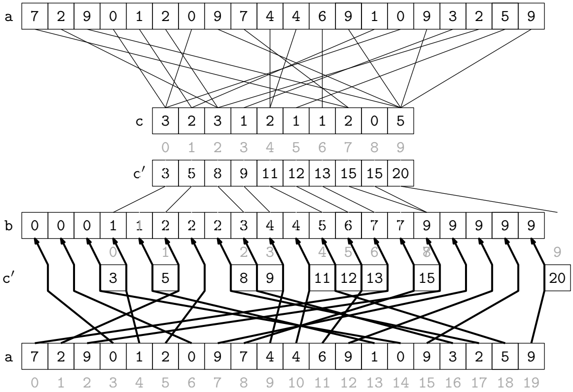
- In-class problem:
  - Design an algorithm that takes an array `a` of `n` integers in the range $\{0,\ldots,\mathtt{k}-1\}$ and sorts them in $O(\mathtt{n}+\mathtt{k})$ time

Counting sort

```
int[] countingSort(int[] a, int k) {
  int c[] = new int[k];
  for (int i = 0; i < a.length; i++)
    c[a[i]]++;
  for (int i = 1; i < k; i++)
    c[i] += c[i-1];
  int b[] = new int[a.length];
  for (int i = a.length-1; i >= 0; i--)
    b[--c[a[i]]] = a[i];
  return b;
}
```

Counting sort



- **Theorem:** The counting sort algorithm can sort an array $\mathtt{a}$ of $\mathtt{n}$ integers in the range $\{0,\ldots,\mathtt{k}-1\}$ in $O(\mathtt{n}+\mathtt{k})$ time

Radix sort

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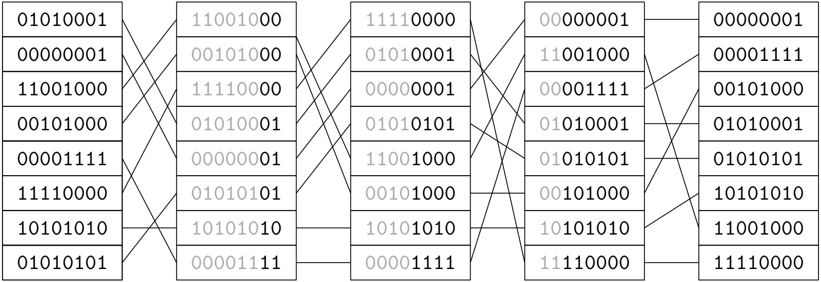
- Correctness depends on fact that counting sort is *stable*
    - if a[i] = a[j] and i < j then a[i] appears before a[j] in the output

Counting sort



01010001	11001000	11110000	00000001	00000001
00000001	00101000	01010001	11001000	00001111
11001000	11110000	00000001	00001111	00101000
00101000	01010001	01010101	01010001	01010001
00001111	00000001	11001000	01010101	01010101
11110000	01010101	00101000	00101000	10101010
10101010	10101010	10101010	10101010	11001000
01010101	00001111	00001111	11110000	11110000

- **Theorem:** The radix-sort algorithm can sort an array $\mathtt{a}$ of $\mathtt{n}$ $\mathtt{w}$-bit integers in $O(\mathtt{n}+2^{\mathtt{d}})$ time

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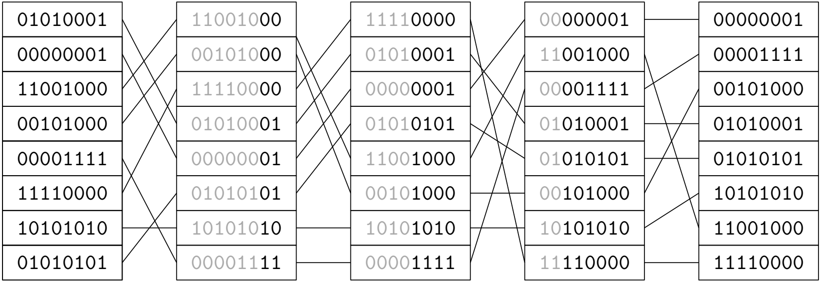

01010001	11001000	11110000	00000001	00000001
00000001	00101000	01010001	11001000	00001111
11001000	11110000	00000001	00001111	00101000
00101000	01010001	01010101	01010001	01010001
00001111	00000001	11001000	01010101	01010101
11110000	01010101	00101000	00101000	10101010
10101010	10101010	10101010	10101010	11001000
01010101	00001111	00001111	11110000	11110000

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- **Theorem:** The radix-sort algorithm can sort an array $\mathtt{a}$ of $\mathtt{n}$ integers in the range $\{0, \ldots, \mathtt{n}^c - 1\}$ in $O(c\mathtt{n})$ time.

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- Quicksort, Heapsort, and Mergesort can each sort an array of length $\mathtt{n}$ in $O(\mathtt{n}\log\mathtt{n})$ time

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    - These work for any `Comparable` data type
    - Quicksort and Heapsort are *in-place* but do more comparisons
    - Mergesort requires an auxiliary array
- Radix-sort can sort an array `a` of `n` integers in the range $\{0,\ldots,\mathtt{n}^c-1\}$ in $O(c\mathtt{n})$ time (and does no comparisons).