Plane Sweep

Pat Morin
COMP2402/2002

Carleton University

Line Segment Intersection Finding

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- Output: All pairs $s, t \in S$ such that s intersects t

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- Then the size of the output is $\binom{n}{2} = \Omega(n^2)$

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- This is much faster when $k \ll \binom{n}{2}$

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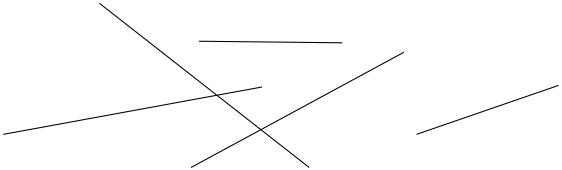

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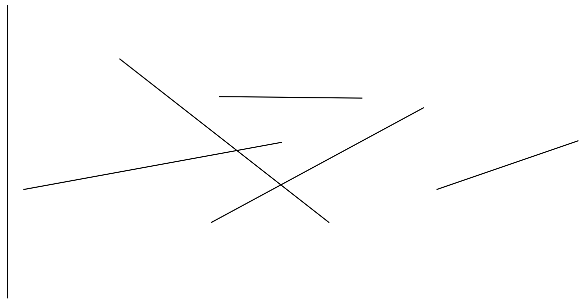

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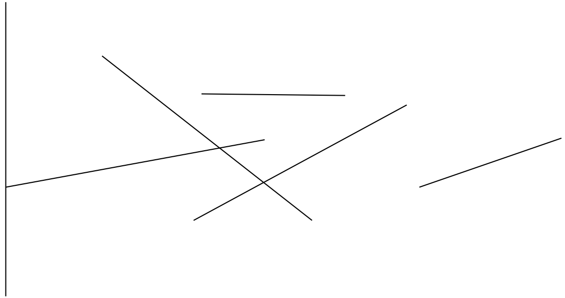

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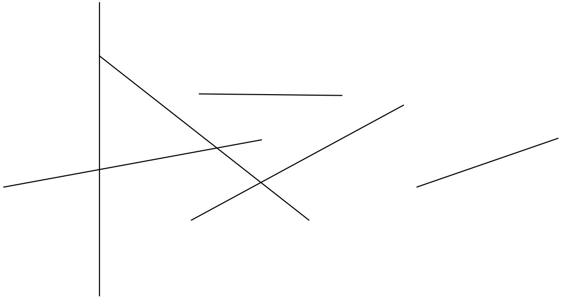

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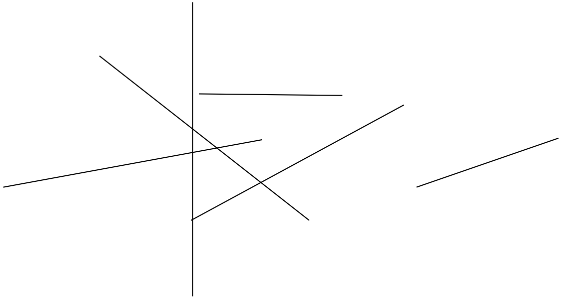

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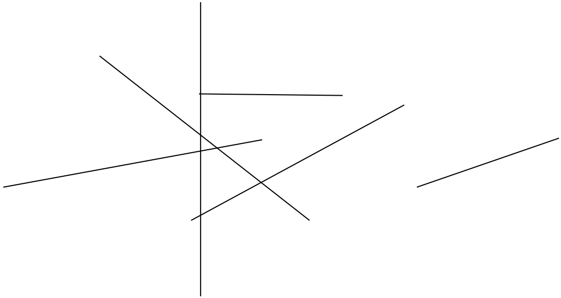

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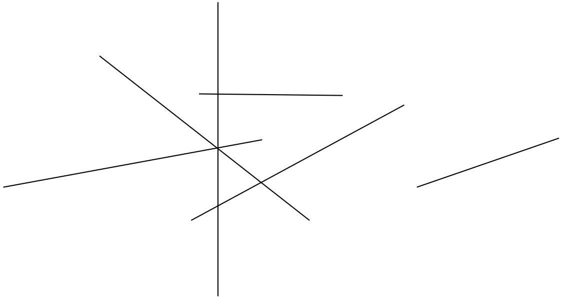

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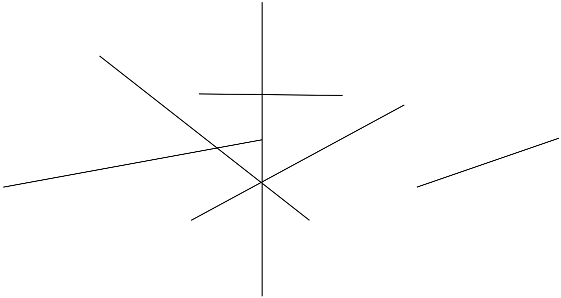

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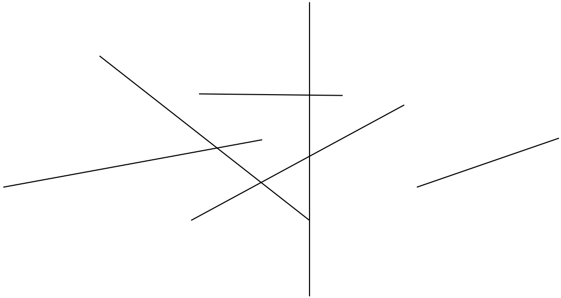

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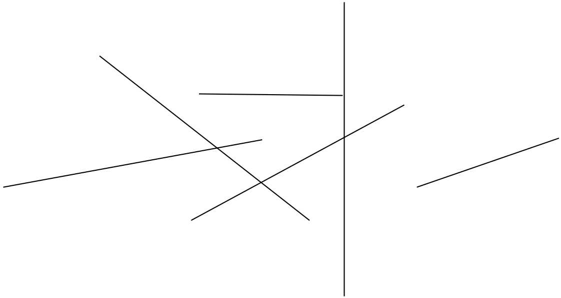

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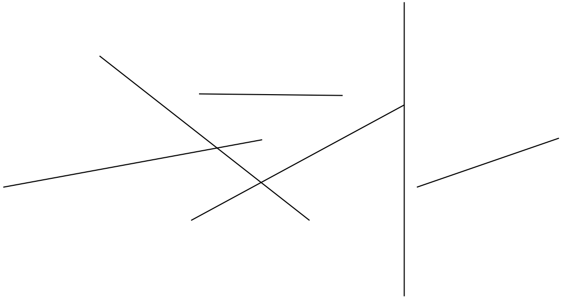

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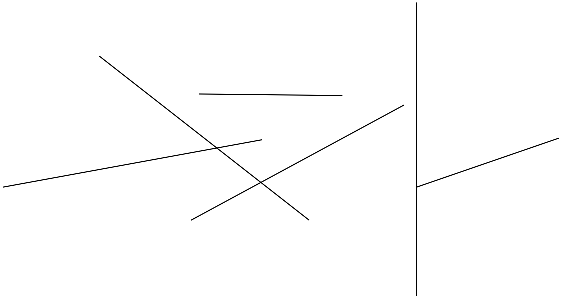

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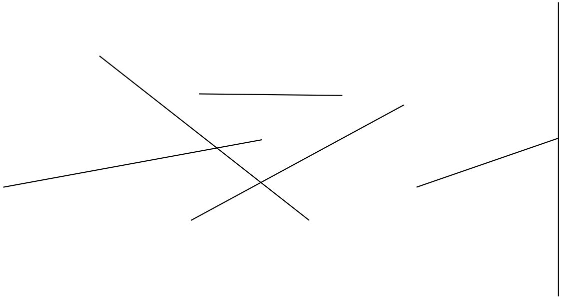

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- The *sweep-line status* is a `SortedSet` that stores the segments that currently intersect the sweep line, ordered from top to bottom (y-coordinate)
- The *event queue* is a `PriorityQueue` that stores events (segment endpoints and intersections) ordered from left to rigth (x-coordinate)

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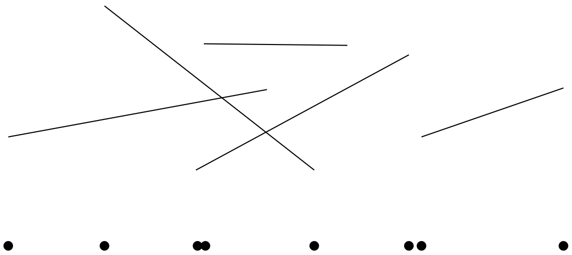

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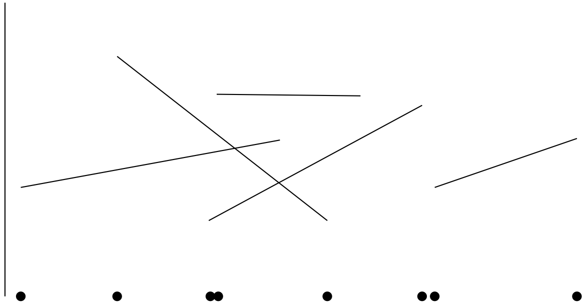

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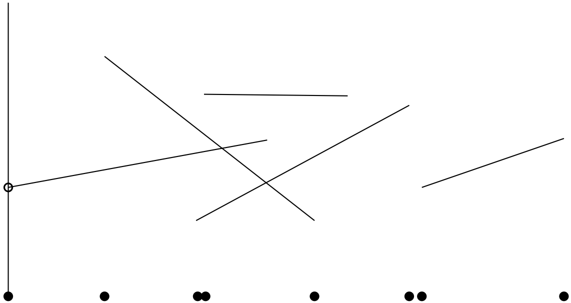

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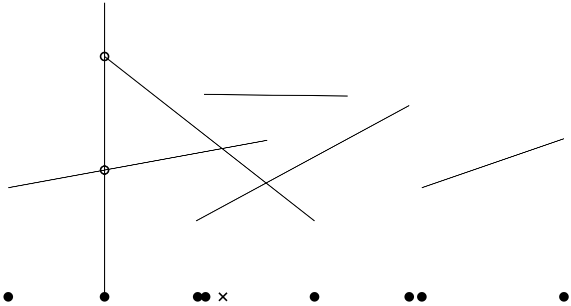

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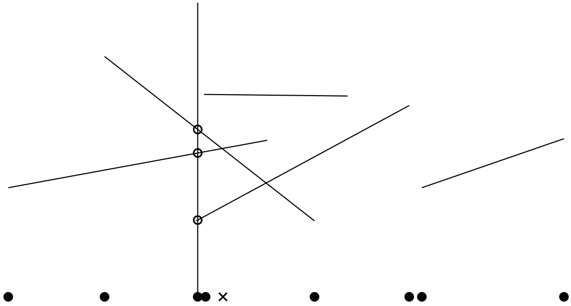

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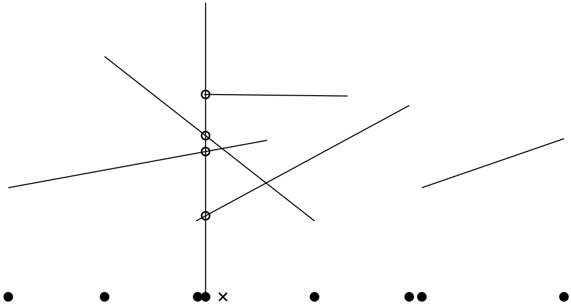

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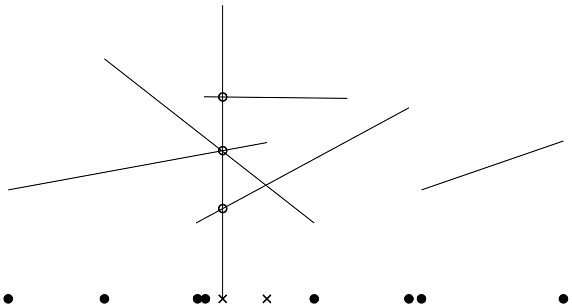

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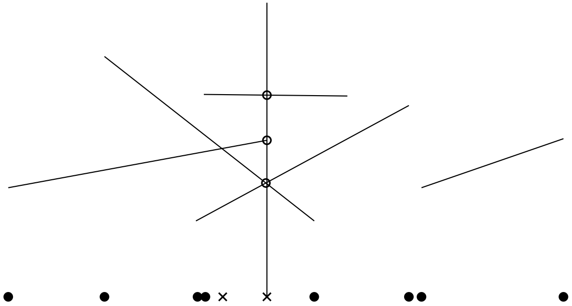

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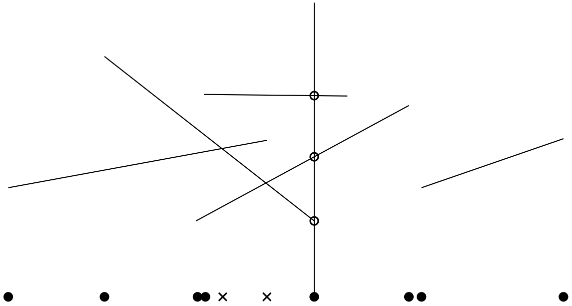

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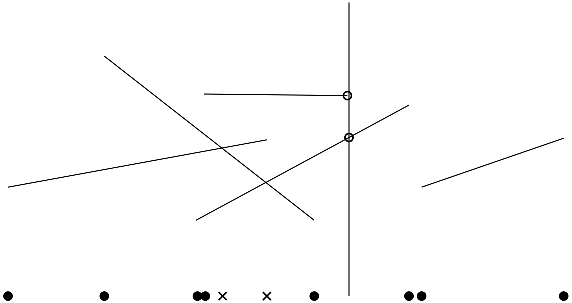

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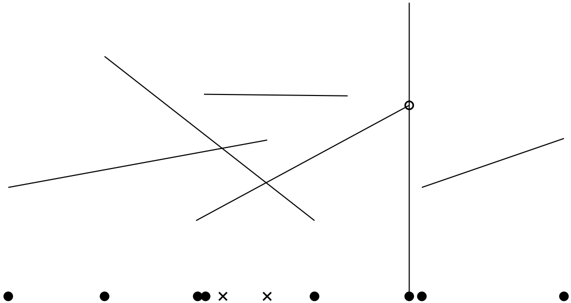

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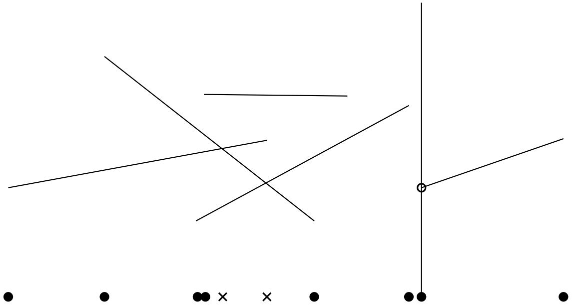

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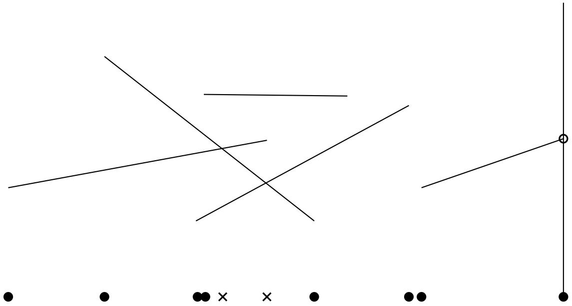

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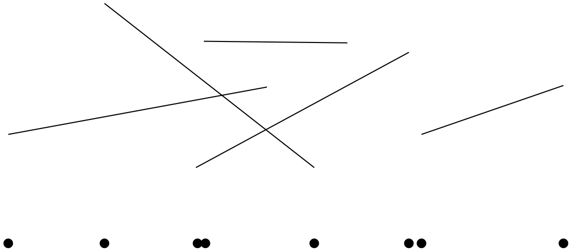

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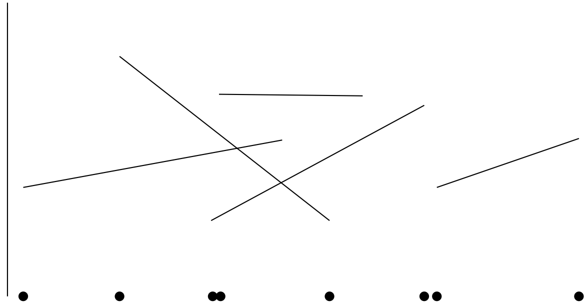

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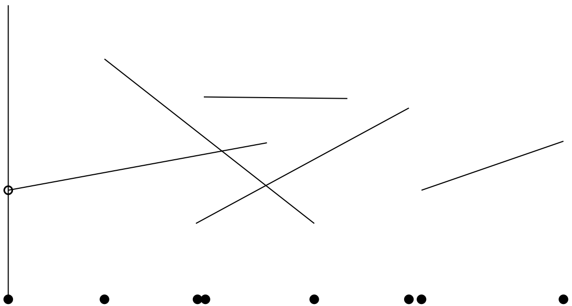

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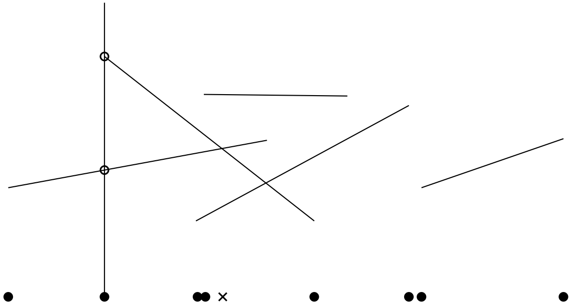

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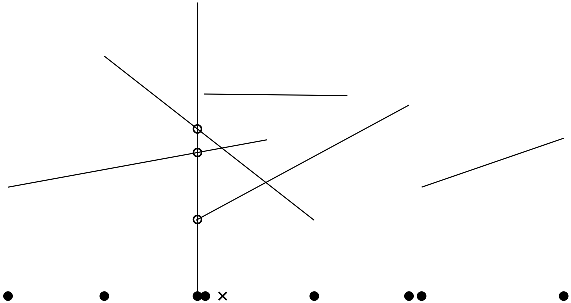

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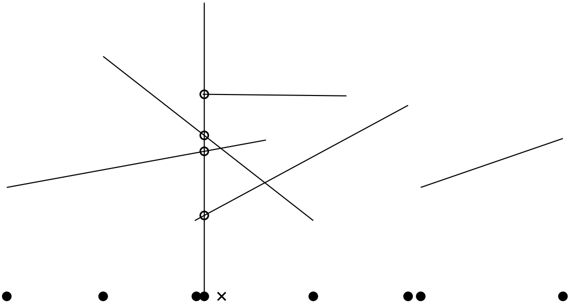

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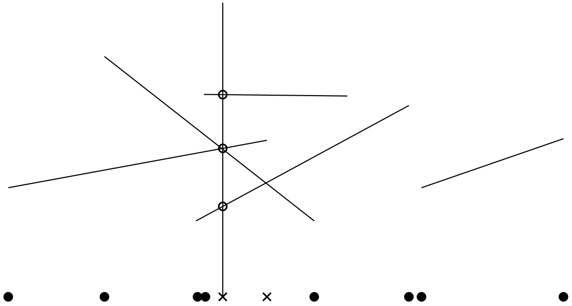

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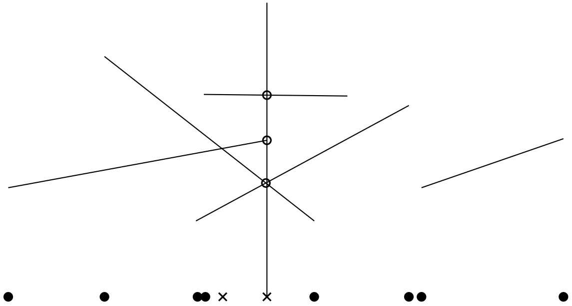

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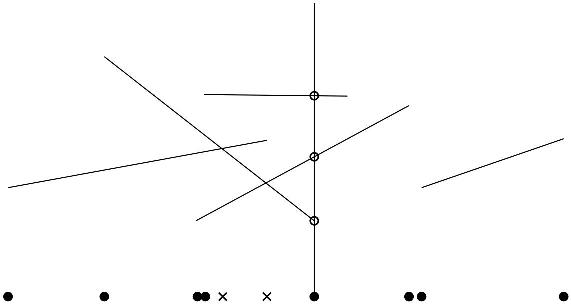

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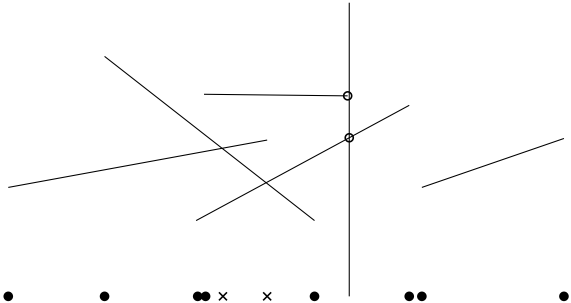

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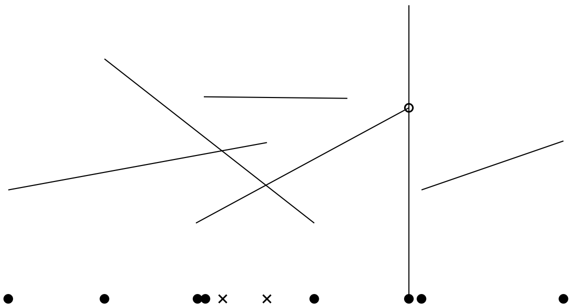

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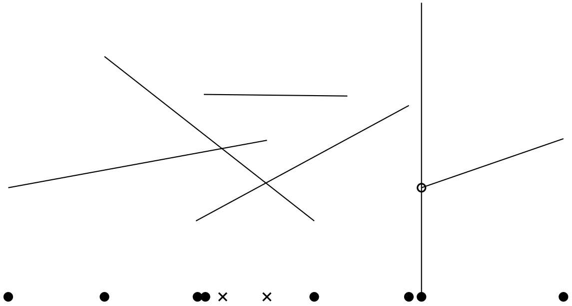

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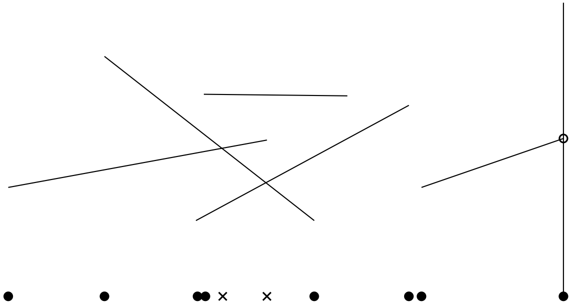

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- Total running time is therefore $(2n + k) \cdot O(\log n) = O((n + k) \log n)$

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  - Given any set of objects, determine if any pair in the set intersect: $O(n \log n)$ time
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  - A data structure for the planar point location problem: $O(n \log n)$ space and $O(\log n)$ query time