Convex Hulls: An Application of Stacks and Deques

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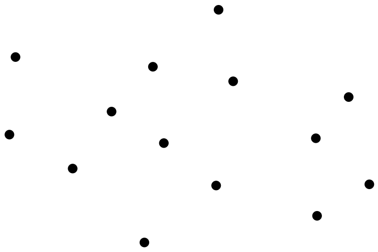
COMP2402/2002

Carleton University

Convex Hulls

- Let $P = \{p_0, \ldots, p_{n-1}\}$ be a set of points in the plane
- The *convex hull* of S
  - the smallest convex set that contains S.



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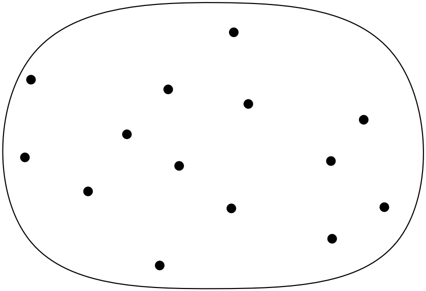

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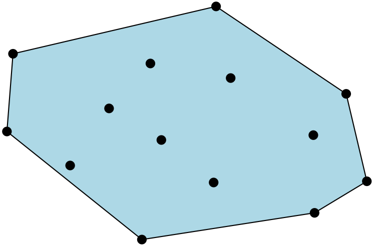

Upper and Lower Hulls

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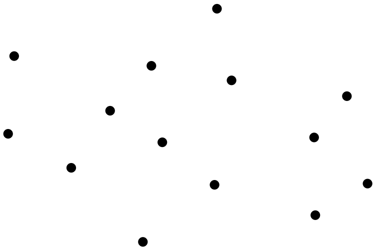

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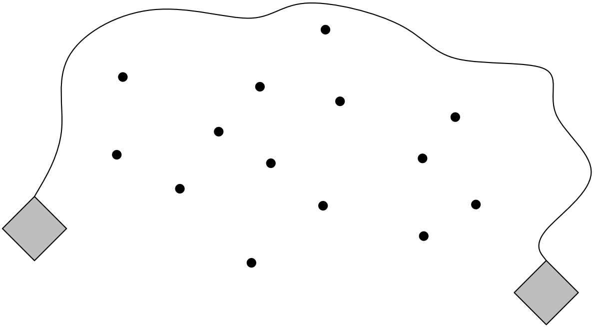

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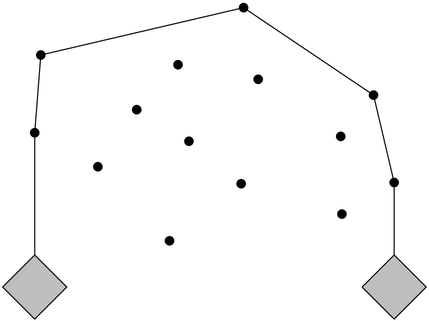

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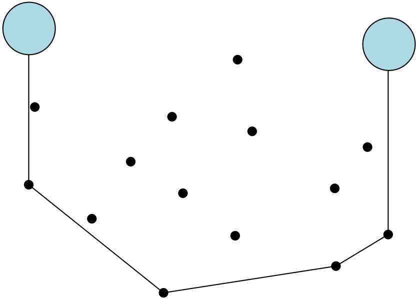

Upper and Lower Hulls

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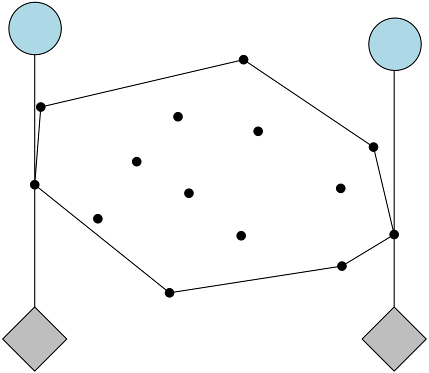
- Convex hull $=$ upper hull $+$ lower hull



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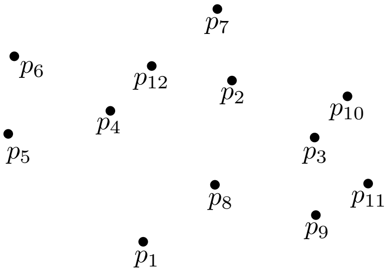
- Input: A set P of n points (unordered)



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- Input: A set P of n points (unordered)
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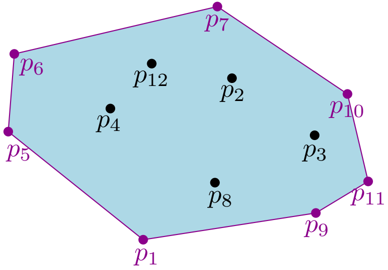
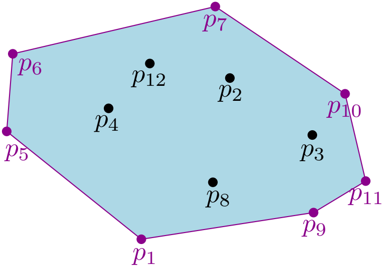


$\langle p_5, p_6, p_7, p_{10}, p_{11}, p_9, p_1 \rangle$

Summary so far

- Input: A set P of n points (unordered)
- Output: A list of the points of P on the convex hull — in clockwise order
- First compute the upper hull, then the lower hull



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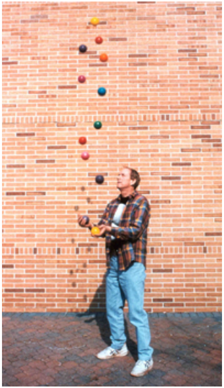
Graham's Scan

- Invented by Ron Graham in 1973



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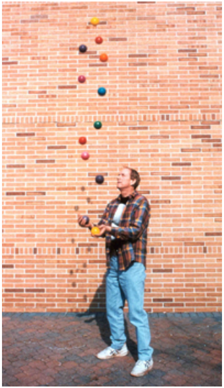

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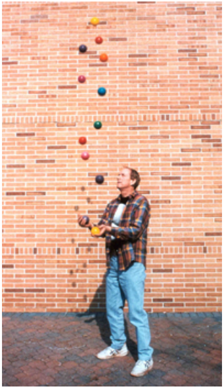
- Invented by Ron Graham in 1973
- Sorts the points by x-coordinate
- Constructs the upper hull incrementally using a stack



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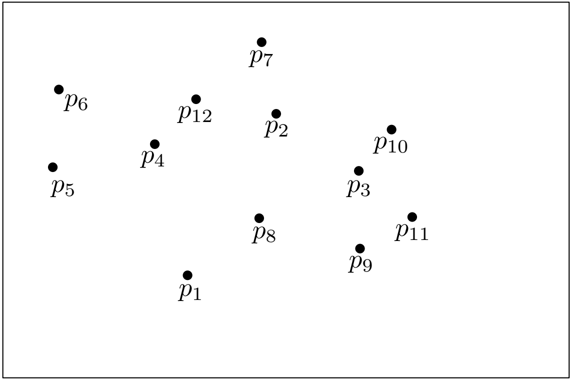

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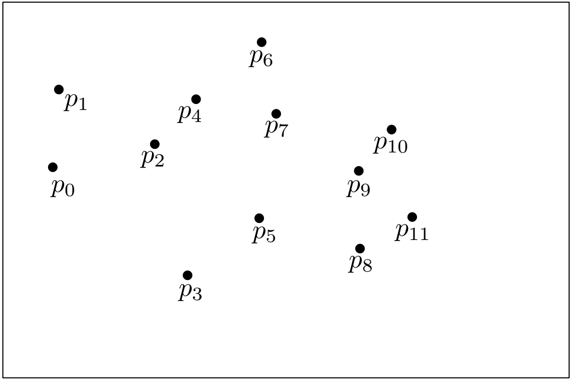

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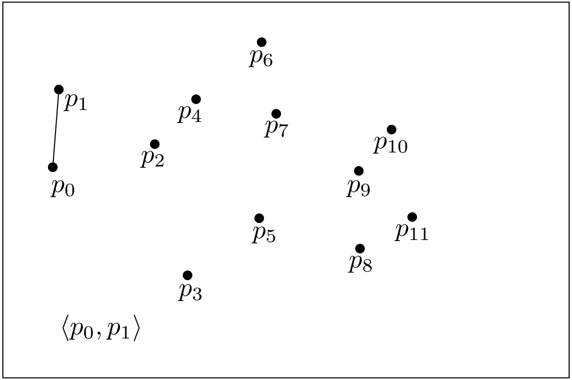

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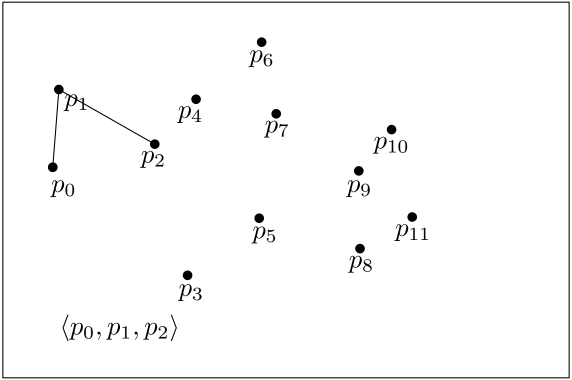

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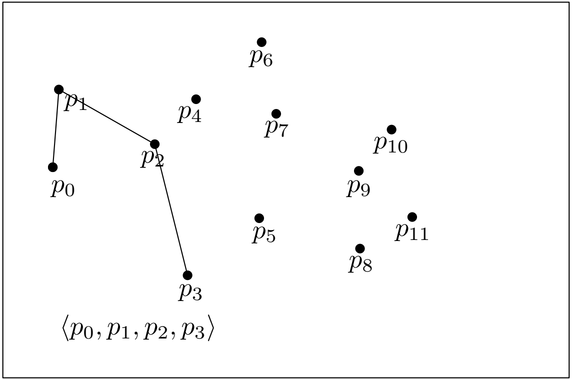

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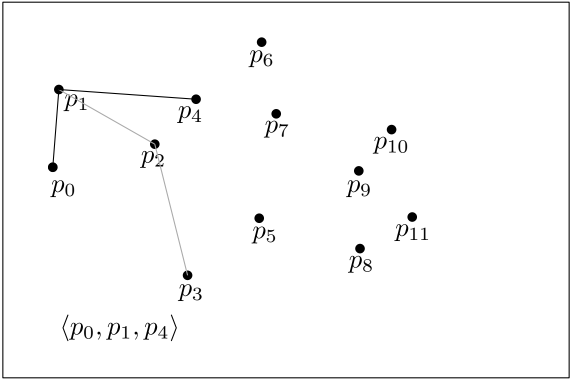

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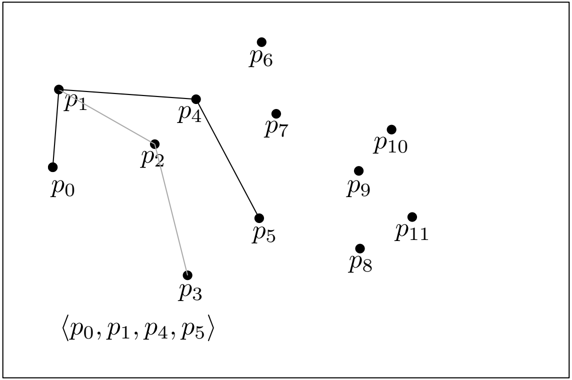

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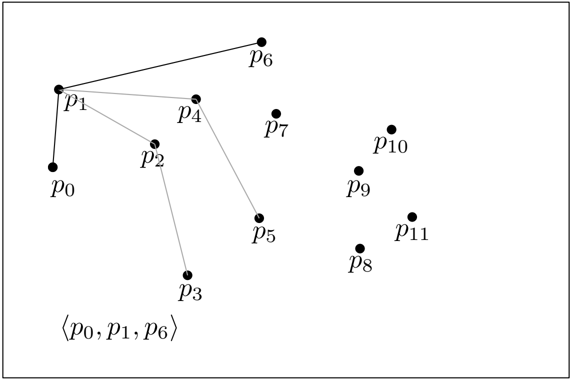

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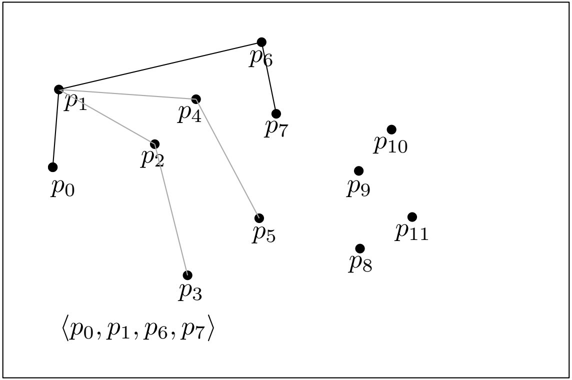

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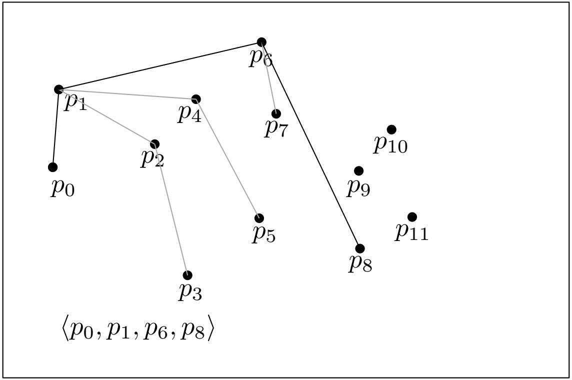

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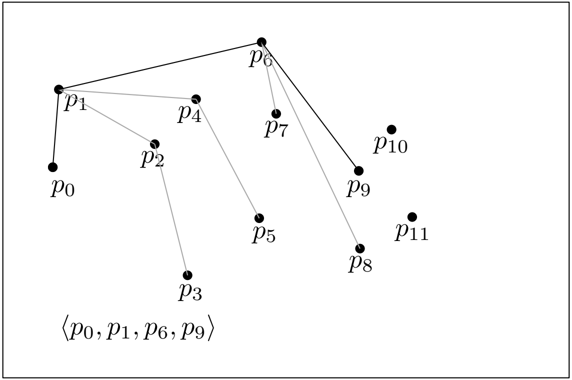

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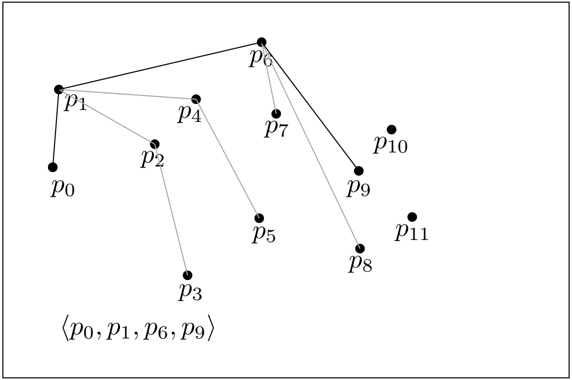

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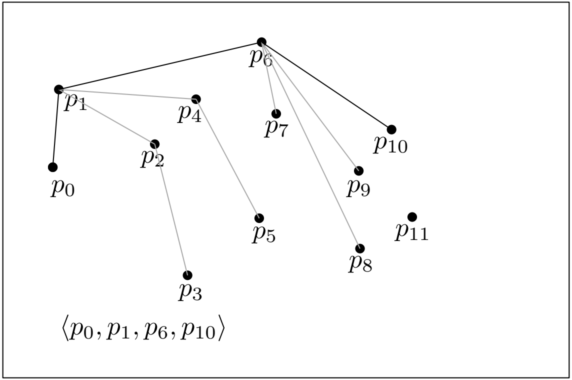

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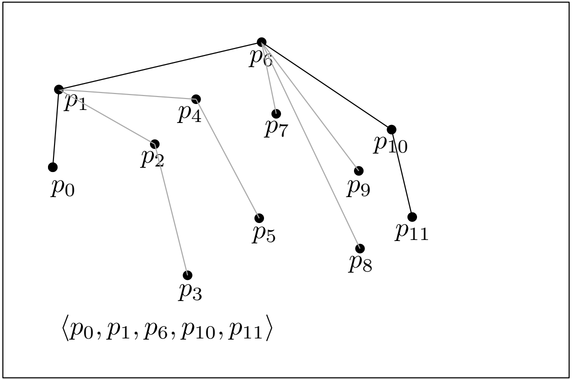

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- Suppose `s` contains the upper hull of $p_0, \ldots, p_{i-1}$
- We want to compute upper hull of $p_0, \ldots, p_i$
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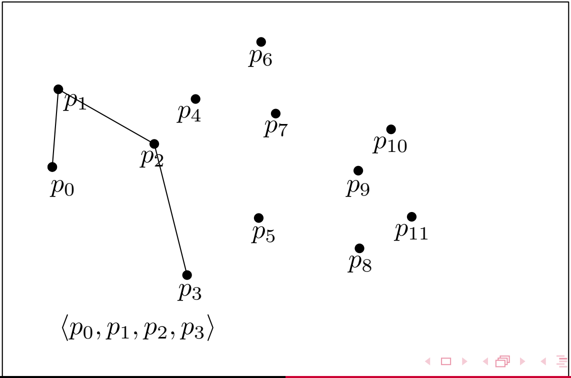

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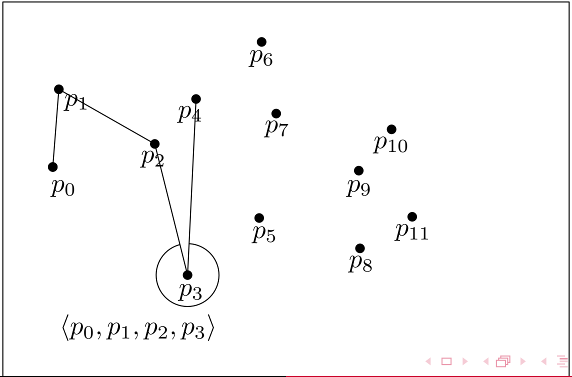

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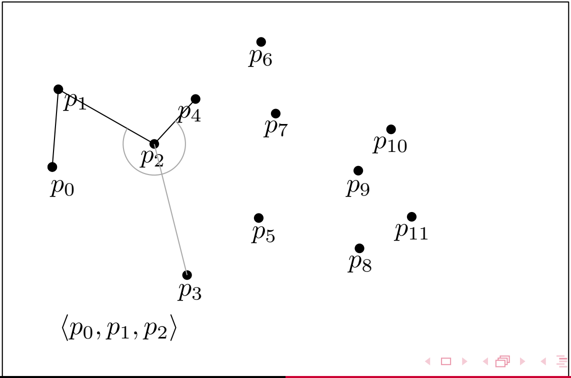

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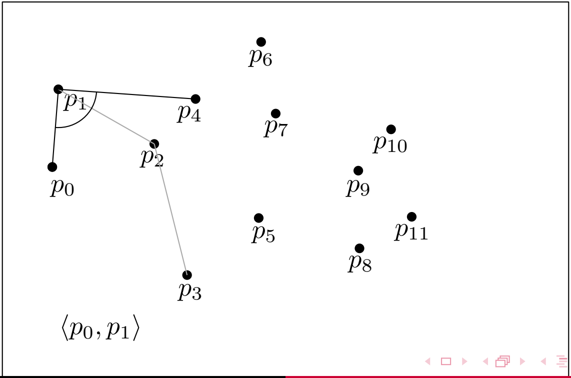

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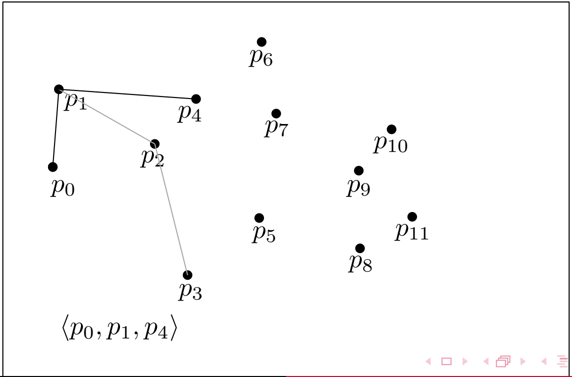

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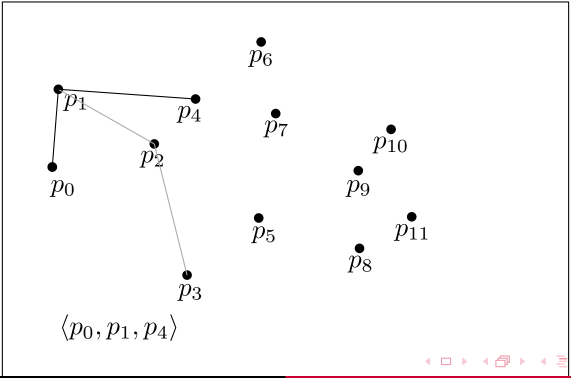

Graham's Scan – in Java

```
public static List<Point2D>
      grahamScan(List<Point2D> p) {
   Collections.sort(p, new XComparator());
   List<Point2D> s = new ArrayList<Point2D>();
   s.add(p.get(0)); s.add(p.get(1));
   for (int i = 2; i < p.size(); i++) {
      Point2D pi = p.get(i);
      while (s.size() >= 2
            && leftTurn(s.get(s.size()-2),
                        s.get(s.size()-1),
                        pi)) {
         s.remove(s.size());    // pop
      }
      s.add(pi);
   }
   return s;
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- **Theorem:** Given a collection P of n points in the plane, two applications of Graham's Scan can compute their convex hull in $O(n \log n)$ time.

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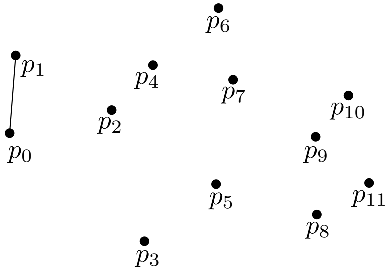
- With two passes, we can compute the upper hull and lower hull and attach them together to get the convex hull:
- **Theorem:** Given a collection P of n points in the plane, two applications of Graham's Scan can compute their convex hull in $O(n \log n)$ time.
- By using a Deque, we only need one pass

Graham's Scan with a deque

- Graham's Scan can compute the convex hull in one-pass using a deque



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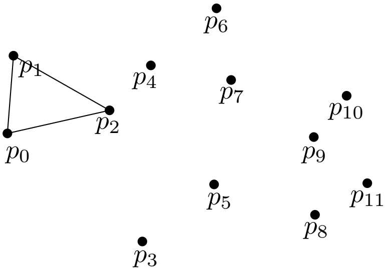

$\langle p_2, p_0, p_1, p_2 \rangle$

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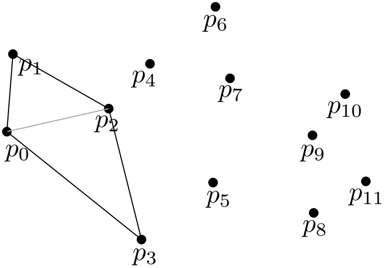

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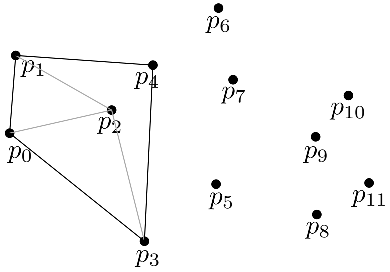

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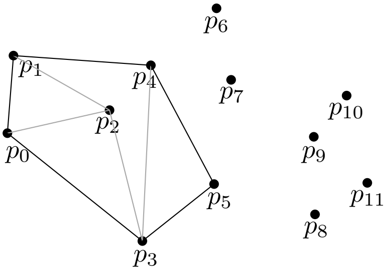

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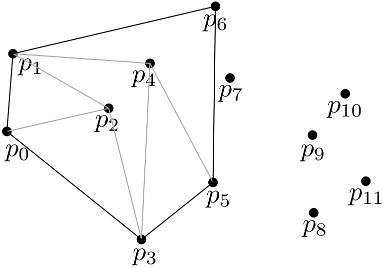

$\langle p_6, p_5, p_3, p_0, p_1, p_6 \rangle$

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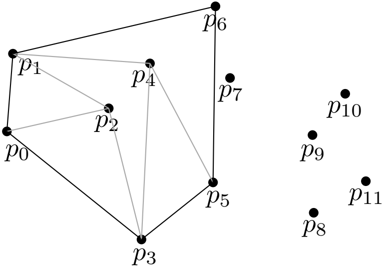
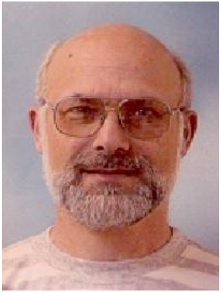
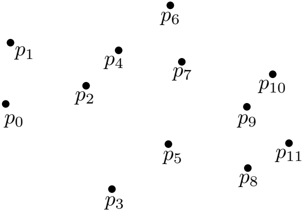


$\langle p_6, p_5, p_3, p_0, p_1, p_6 \rangle \ \ldots$

Melkman's Algorithm

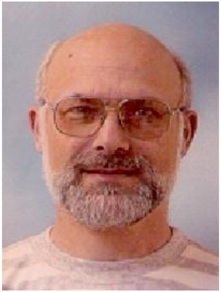
- Graham's Scan starts by sorting the the points by x-coordinate

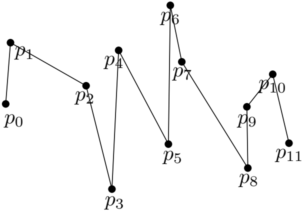




Melkman's Algorithm

- Graham's Scan starts by sorting the the points by x-coordinate
  - This means that $p_0, \ldots, p_{n-1}$ becomes a non-self-intersecting path
  - If points are already sorted then Graham's Scan takes $O(n)$ time





Melkman's Algorithm

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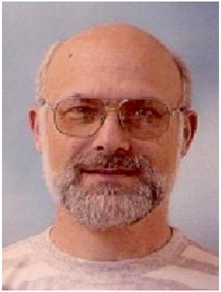

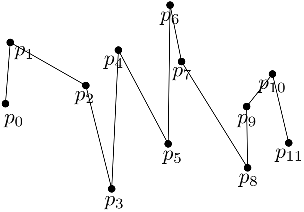

Melkman's Algorithm

- Graham's Scan starts by sorting the the points by x-coordinate
  - This means that $p_0, \ldots, p_{n-1}$ becomes a non-self-intersecting path
  - If points are already sorted then Graham's Scan takes $O(n)$ time
- Melkman's Algorithm:
  - Works for any non-self-intersecting path $p_0, \ldots, p_{n-1}$