# Simultaneous Diagonal Flips in Plane Triangulations *

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**Abstract**

Simultaneous diagonal flips in plane triangulations are investigated. It is proved that every triangulation with at least six vertices has a simultaneous flip into a 4-connected triangulation. This result is used to prove that for any two $n$-vertex triangulations, there exists a sequence of $O(\log n)$ simultaneous flips to transform one into the other. The maximum size of a simultaneous flip is then studied. It is proved that every simultaneous flip has at most $n-2$ edges, and there exist triangulations with a maximum simultaneous flip of $\frac{6}{7}(n-2)$ edges. On the other hand, it is shown that every triangulation has a simultaneous flip of at least $\frac{2}{3}(n-2)$ edges.

## 1 Introduction

Let $G=(V,E)$ be a *triangulation*; that is, a simple planar graph with a fixed plane (combinatorial) embedding such that every face consists of three edges. Let $vw$ be an edge of $G$. Let $x$ and $y$ be the vertices such that $vw$ is incident to the faces $vwx$ and $wvy$. If $xy$ is not an edge of $G$ then $vw$ is *flippable*. Suppose $vw$ is flippable. Let $G'$ be the triangulation obtained from $G$ by deleting $vw$ and adding the edge $xy$, such that in the cyclic order of edges incident to $x$ (respectively, $y$), $xy$ is added between $xv$ and $xw$ ($yw$ and $yv$). We say $G$ is *flipped* into $G'$ by $vw$. This operation is called a (*diagonal*) *flip*, and is illustrated in Figure 1(a).

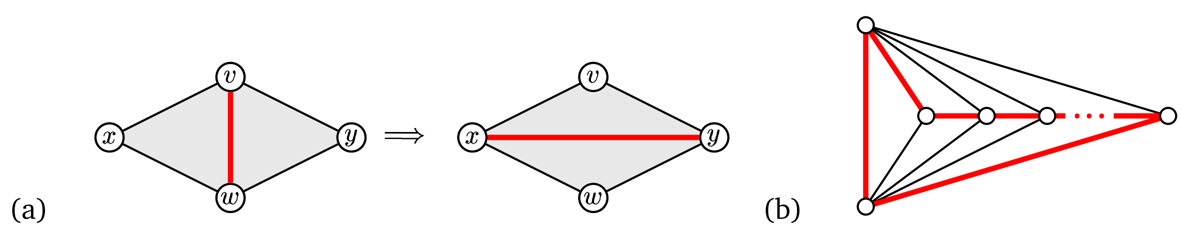


Figure 1: (a) Edge $vw$ is flipped into $xy$. (b) The canonical triangulation and a Hamiltonian cycle.

Diagonal flips in planar triangulations, and on other surfaces, are widely studied (see the survey by Negami [7]). Wagner [11] proved that there is finite sequence of diagonal flips to transform a given triangulation into any other with the same number of vertices. In particular, it is proved that any triangulation can be transformed into the triangulation shown in Figure 1(b).

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Komuro [6] proved that in fact $O(n)$ flips suffice, and Gao *et al.* [5] proved that $O(n \log n)$ flips suffice for labelled triangulations.

Two edges of a triangulation that are incident to a common face are *consecutive*. To simultaneously flip two edges in a triangulation it is necessary that they are not consecutive. Let $S$ be a set of edges in a triangulation $G$ such that $S$ contains no two consecutive edges, and no parallel edges are produced by flipping every edge in $S$. Then we say $S$ is (*simultaneously*) *flippable*. Note that it is possible for $S$ to be flippable, yet $S$ contains non-flippable edges, and it is possible for every edge in $S$ to be flippable, yet $S$ itself is not flippable. The graph obtained from a triangulation $G$ by flipping every edge in a flippable set $S$ is denoted by $G\langle S\rangle$. We say $G$ is *flipped* into $G\langle S\rangle$ by $S$. This operation is called a *simultaneous* (*diagonal*) *flip*. As far as the authors are aware, simultaneous flips have only been studied in the more restrictive context of geometric triangulations of a point set [4].

Our first result states that every triangulation can be transformed by one simultaneous flip into a *Hamiltonian* triangulation; that is a triangulation containing a spanning cycle. This result is presented in Section 2. In Section 3 we prove that for any two $n$-vertex triangulations, there exists a sequence of $O(\log n)$ simultaneous flips to transform one into the other. This result is optimal for many pairs of triangulations. For example, if one triangulation has $\Theta(n)$ maximum degree and the other has $O(1)$ maximum degree, then $\Omega(\log n)$ simultaneous flips are needed, since one simultaneous flip can at most halve the degree of a vertex. This is also holds for diameter instead of maximum degree. Finally in Section 4 the maximum size of a simultaneous flip is studied. It is proved that every simultaneous flip has at most $n-2$ edges, and there exist triangulations with a maximum simultaneous flip of $\frac{6}{7}(n-2)$ edges. On the other hand, it is shown that every triangulation has a simultaneous flip of $\frac{2}{3}(n-2)$ edges.

## 2 Flipping into a Hamiltonian Triangulation

We have the following sufficient condition for a set of edges to be flippable. A cycle $C$ in a triangulation $G$ is *separating* if deleting the vertices of $C$ from $G$ produces a disconnected graph. A 3-cycle in a triangulation is called a *triangle*.

**Lemma 1.** *Let $S$ be a set of edges in a triangulation $G$ such that no two edges of $S$ appear in a common triangle, and every edge in $S$ is in a separating triangle. Then $S$ is flippable.*

*Proof.* Since no two edges of $S$ appear in the same triangle, no two edges in $S$ are consecutive. Suppose, for the sake of contradiction, that $G\langle S\rangle$ has parallel edges $e_1 = vw$ and $e_2 = vw$. Let $S'$ be the set of edges in $G\langle S\rangle$ that are not in $G$. Initially suppose that exactly one of $e_1$ and $e_2$, say $e_1$, is in $S'$; see Figure 2(a). Let $xy$ be the edge of $G$ flipped to $e_1$. By assumption, $xy$ is in a separating triangle $xyz$ of $G$. Then in $G$, $z$ is either inside the triangle $\{e_2, vx, xw\}$ or outside the triangle $\{vy, yw, e_2\}$. ($z \neq v$ and $z \neq w$ as otherwise $xyz$ would not be a separating triangle.) However, in these cases, $z$ could not be adjacent to $y$ and $x$, respectively, without breaking planarity. Now suppose that $e_1$ and $e_2$ are both in $S'$; see Figure 2(b). Let $xy$ be the edge of $G$ flipped to $e_1$, and let $rs$ be the edge of $G$ flipped to $e_2$. By assumption, each of $xy$ and $rs$ is in a separating triangle of $G$. Let $z$ and $t$ be the vertices of $G$ such that $xyz$ and $rst$ are separating triangles. For $xyz$ and $rst$ to be separating triangles of $G$, it must be the case that $z = r$, $t = x$, and $y = s$; or $z = s$, $t = y$, and $x = r$. In either case, $xy$ and $rs$ are in a common triangle in $G$, contradicting the initial assumption. In each case, we have a contradiction. Hence $S$ is flippable. □

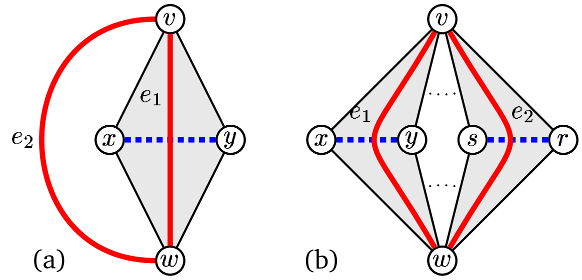


Figure 2: Dashed edges are flipped to create bold parallel edges. Shaded regions are faces.

**Lemma 2.** *Let $G$ be a triangulation with $n \geq 6$ vertices. Let $S$ be a set of edges in $G$ that satisfy Lemma 1 such that every separating triangle contains an edge in $S$. Then $G\langle S\rangle$ is 4-connected.*

*Proof.* Suppose for the sake of contradiction, that $G\langle S\rangle$ contains a separating triangle $T = uvw$. Let $S'$ be the set of edges in $G\langle S\rangle$ that are not in $G$. We proceed by case-analysis on $|T \cap S'|$ (refer to Figure 3). Since every separating triangle in $G$ has an edge in $S$, $|T \cap S'| \geq 1$.

Case 1. $|T \cap S'| = 1$: Without loss of generality, $vw \in S'$, $uv \notin S'$, and $uw \notin S'$. Suppose $xy$ was flipped to $vw$. Then $xy$ is in a separating triangle $xyp$ in $G$. By an argument similar to that in Lemma 1, $p = u$. Since $G$ has at least six vertices, at least one of the triangles $\{uvx, uvy, uwx, uwy\}$ is a separating triangle. Thus at least one of the edges in these triangles is in $S$. Since $xy \in S$, and no two edges of $S$ appear in a common triangle, $\{ux, uy, vx, vy, wx, wy\} \cap S = \emptyset$. Thus $uv$ or $uw$ is in $S$. But then $uvw$ is not a triangle in $G\langle S\rangle$, which is a contradiction.

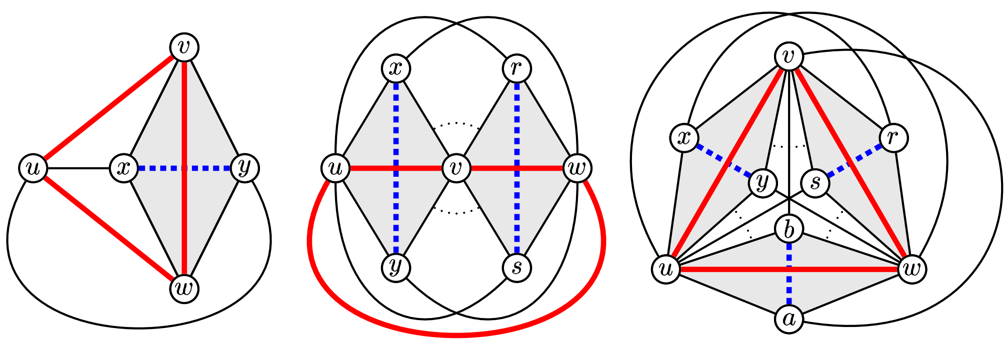


Figure 3: Dashed edges are flipped to create a bold separating triangle. Shaded regions are faces.

Case 2. $|T \cap S'| = 2$: Without loss of generality, $uv \in S'$, $vw \in S'$, and $uw \notin S'$. Suppose $xy$ was flipped to $uv$, and $rs$ was flipped to $vw$. Without loss of generality, $y$ and $s$ are inside $uvw$ in $G\langle S\rangle$. Then in $G$, $xy$ was in a separating triangle $xyz$, and $rs$ was in a separating triangle $rst$. By an argument similar to that in Lemma 1, $z = w$ and $t = u$. But then the subgraph of $G$ induced by $\{u, v, w, x, y, r, s\}$ is not planar, or it contains parallel edges in the case that $x = r$ and $y = s$.

Case 3. $|T \cap S'| = 3$: Suppose $xy$ was flipped to $uv$, $rs$ was flipped to $vw$, and $ab$ was flipped to $uw$. Without loss of generality, $y$, $s$ and $b$ are inside $uvw$ in $G\langle S\rangle$. In $G$, $xy$ was in a separating triangle $xyz$, $rs$ was in a separating triangle $rst$, and $ab$ was in a separating triangle $abc$. By an argument similar to that in Lemma 1, $z = w$, $t = u$, and $c = v$. But then the subgraph of $G$ induced by $\{u, v, w, x, y, r, s, a, b\}$ is non-planar, or contains parallel edges in the case that $y = s = b$ and $x = r = a$.

In each case we have derived a contradiction. Thus $G\langle S\rangle$ has no separating triangle. $\square$

Observe that the restriction in Lemma 2 to triangulations with at least six vertices is unavoidable. Every triangulation with at most five vertices has a vertex of degree three, and is thus not 4-connected. We now turn to the question of determining a set of flippable edges. Consider the following consequence of Petersen's matching theorem.

**Lemma 3.** *Let $e$ be an edge of an $n$-vertex triangulation $G$. Then $G$ has a set of edges $S$ that can be computed in $O(n)$ time, such that $e \in S$ and every triangle of $G$ has exactly one edge in $S$.*

*Proof.* We proceed by induction on the number of separating triangles. Suppose $G$ is a triangulation with no separating triangles. Then the dual $G^*$ is a 3-regular bridgeless planar graph. Biedl *et al.* [1] prove the following strengthening of Petersen's matching theorem [8]: For every 3-regular bridgeless planar graph $H$, and for every edge $f$ of $H$, there is a perfect matching of $H$ containing $f$ that can be computed in $O(n)$ time. Applying this result with $H = G^*$ and $f = e$, we obtain a perfect matching of $G^*$ containing $e$, which corresponds to a set of edges $S$ in $G$ with $e \in S$ such that every face of $G$ has exactly one edge in $S$. Since $G$ has no separating triangles, $S$ has exactly one edge for every triangle of $G$.

Now suppose $G$ has $k > 0$ separating triangles, and the lemma holds for triangulations with less than $k$ separating triangles. Let $T$ be a separating triangle of $G$. Let the components of $G \setminus T$ have vertex sets $V_1$ and $V_2$. Consider the induced subgraphs $G_1 = G[V_1 \cup T]$ and $G_2 = G[V_2 \cup T]$. Without loss of generality, suppose the given edge $e$ is in $G_1$. Both $G_1$ and $G_2$ have less than $k$ separating triangles. By induction $G_1$ has a set of edges $S_1$ such that $e \in S_1$, and every triangle of $G_1$ has exactly one edge in $S_1$. Let $e_2$ be the edge in $S_1 \cap T$. By induction, $G_2$ has a set of edges $S_2$ such that $e_2 \in S_2$, and every triangle of $G_2$ has exactly one edge in $S_2$. Thus $S = S_1 \cup S_2$ is a set of edges of $G$ such that $e \in S$, and every triangle of $G$ has exactly one edge in $S$.

A modification of the algorithm of Biedl *et al.* [1] computes the set $S$ in $O(n)$ time (see [2]). $\square$

**Theorem 1.** *Every triangulation $G$ with $n \geq 6$ vertices has a simultaneous flip into a 4-connected triangulation that can be computed in $O(n)$ time.*

*Proof.* By Lemma 3, $G$ has a set of edges $S_0$ such that every triangle of $G$ has exactly one edge in $S_0$. The set $S$ of edges in $S_0$ that are in a separating triangle of $G$ can be computed in $O(n)$ time as a by-product of the algorithm mentioned in Lemma 3. By Lemma 1, $S$ is flippable. By Lemma 2, $G\langle S\rangle$ is 4-connected. $\square$

We can obtain a stronger result at the expense of a slower algorithm. The following consequence of the 4-colour theorem is essentially a Tait edge-colouring [10].

**Lemma 4.** *Every $n$-vertex planar graph $G$ has an edge 3-colouring that can be computed in $O(n^2)$ time, such that every triangle receives three distinct colours.*

*Proof.* Robertson *et al.* [9] prove that $G$ has a proper vertex 4-colouring that can be computed in $O(n^2)$ time. Let the colours be $\{1, 2, 3, 4\}$. Colour an edge *red* if its end-points are coloured 1 and 2, or 3 and 4. Colour an edge *blue* if its end-points are coloured 1 and 3, or 2 and 4. Colour an edge *green* if its end-points are coloured 1 and 4, or 2 and 3. Since each triangle $T$ of $G$ receives three distinct vertex colours, $T$ also receives three distinct edge colours. $\square$

**Theorem 2.** *Let $G$ be a triangulation with $n \geq 6$ vertices. Then $G$ has three pairwise disjoint flippable sets of edges $S_1, S_2, S_3$ that can be computed in $O(n^2)$ time, such that each $G\langle S_i\rangle$ is 4-connected.*

*Proof.* By Lemma 4, $G$ has an edge 3-colouring such that every triangle receives three distinct colours. For any of the three colours, let $S$ be the set of edges receiving that colour and in a separating triangle of $G$. By Lemma 1, $S$ is flippable. By Lemma 2, $G\langle S\rangle$ is 4-connected. □

We have the following corollary of Theorems 1 and 2, since every triangulation on at most five vertices (that is, $K_3$, $K_4$ or $K_5 \setminus e$) is Hamiltonian, and every 4-connected triangulation has a Hamiltonian cycle [12] that can be computed in $O(n)$ time [3].

**Corollary 1.** *Every $n$-vertex triangulation $G$ has a simultaneous flip into a Hamiltonian triangulation that can be computed in $O(n)$ time. Furthermore, $G$ has three disjoint simultaneous flips that can be computed in $O(n^2)$ time, such that each transforms $G$ into a Hamiltonian triangulation.* □

# 3 Simultaneous Flips Between Given Triangulations

In this section we prove the following theorem.

**Theorem 3.** *Let $G_1$ and $G_2$ be triangulations on $n$ vertices. There is a sequence of $O(\log n)$ simultaneous flips to transform $G_1$ into $G_2$.*

The main idea in the proof of Theorem 3 is that $G_1$ and $G_2$ can each be transformed by $O(\log n)$ simultaneous flips into the canonical triangulation shown in Figure 1(b). Thus to transform $G_1$ into $G_2$ we first transform $G_1$ into the canonical triangulation, and then apply the corresponding flips for $G_2$ in reverse order. To transform a given triangulation into the canonical triangulation we first apply Corollary 1 to obtain a Hamiltonian triangulation with one simultaneous flip. We thus have two outerplanar graphs whose intersection is the Hamiltonian cycle. With $O(\log n)$ simultaneous flips we transform each of these outerplanar graphs so that the internal edges of each graph form a star rooted at adjacent vertices. This is the canonical triangulation. Since outerplanar graphs play an important role in the proof of Theorem 3, we begin by outlining some of their properties.

## 3.1 Outerplanar graphs

An outerplanar graph is a plane graph where every vertex lies on a specified face, called the *outerface*. An outerplanar graph is *maximal* if every face is a triangle, except for possibly the outerface. In the remainder of this section, all outerplanar graphs considered are maximal. An edge on the outerface of an outerplanar graph is called *external*, and an edge that is not on the outerface is *internal*. It is well known that a maximal outerplanar graph with $n$ vertices has $2n-3$ edges, and is 3-colourable. Let $G = (V, E)$ be a maximal outerplanar graph. The (*weak*) *dual* $G^*$ of $G$ is defined as follows. Each internal face of $G$ is a vertex of $G^*$ and two vertices of $G^*$ are adjacent if the corresponding faces in $G$ share an edge. This definition of the dual deviates slightly from the standard definition since the outerface is not represented by a vertex. Observe that $G^*$ is a tree with maximum degree at most three. The notions of *diagonal flip* and *flippable set* for triangulations are extended to maximal outerplanar graphs in the natural way, except that only internal edges are allowed to be flipped.

**Lemma 5.** *Every internal edge of a maximal outerplanar graph $G$ is flippable.*

*Proof.* Suppose that an internal edge $e$ is not flippable; that is, flipping $e$ to an edge $e'$ results in a parallel edge. Then $e'$ is already in $G$. But this is a contradiction, since the outerface along with $e$ and $e'$ form a subdivision of $K_4$. □

The next result follows immediately from Lemma 5.

**Lemma 6.** *Let $S$ be a set of internal edges in a maximal outerplanar graph $G$. Then $S$ is flippable if and only if the corresponding dual edges $S^*$ form a matching in $G^*$.* □

**Lemma 7.** *Any set $S$ of internal edges in a maximal outerplanar graph $G$ can be removed from $G$ with three simultaneous flips.*

*Proof.* By the construction used in Lemma 4 we obtain an edge 3-colouring of $G$ such that each internal face is trichromatic. (Note that the 4-colour theorem is not needed here since $G$ is 3-colourable.) Suppose the edge colours are red, blue, and green. Each colour class corresponds to a matching in the dual. By Lemma 6, the red edges in $S$ are flippable. Let $G'$ be the maximal outerplanar graph obtained by flipping the red edges in $S$. Blue edges might be consecutive in $G'$, and similarly for green edges. However, the subgraph of the dual induced by the blue and green edges has maximum degree at most two (since $G^*$ has maximum degree at most three), and thus consists of a set of disjoint paths (since $G^*$ is a tree). Thus two simultaneous flips are sufficient to flip the red and blue edges in $S$. Clearly no edge in $S$ is flipped back into the graph obtained. □

Recall that our aim is to transform a maximal outerplanar graph into one in which the internal edges form a star. The next lemma shows how to do this.

**Lemma 8.** *Let $v$ be a vertex of a maximal outerplanar graph $G$ whose dual tree has diameter $k$. Then $G$ can be transformed by at most $k$ simultaneous flips into a maximal outerplanar graph in which $v$ is adjacent to every other vertex.*

*Proof.* Let $P$ be the set of internal faces incident with $v$ in $G$. In the dual $G^*$, the corresponding vertices of $P$ forms a path $P^*$. Define the *distance* of each vertex $x$ in $G^*$ as the minimum number of edges in a path from $x$ to a vertex in $P^*$. Since the diameter of the dual tree is $k$, every vertex in $G^*$ has distance at most $k$. No two vertices in $G^*$ both with distance one are adjacent, as otherwise $G^*$ would contain a cycle. Each vertex of $P^*$ is adjacent to at most one vertex at distance one, since the endpoints of $P^*$ correspond to faces with an edge on the outerface. Therefore the set of edges incident to $P^*$ but not in $P^*$ form a matching between the vertices at distance one and the vertices of $P^*$, such that all vertices at distance one are matched. By Lemma 6, these edges can be flipped simultaneously. After doing so, the distance of each vertex not adjacent to $P^*$ is reduced by one. Thus, by induction, at most $k$ simultaneous flips are required to reduce the distance of every vertex in $G^*$ to zero, in which case $v$ is adjacent to every other vertex of $G$. □

Lemma 8 suggests that reducing the diameter of the dual tree is a good strategy. In what follows all logarithms are base 2.

**Lemma 9.** *Every maximal outerplanar graph with $n$ vertices can be transformed by a sequence of at most $3\log n \,/\, (\log 3 - 1)$ simultaneous flips into a maximal outerplanar graph whose dual tree has diameter at most $2\log n \,/\, (\log 3 - 1)$.*

*Proof.* We proceed by induction on $n$. The lemma holds trivially for $n = 3$. Assume the lemma holds for all outerplanar graphs with less than $n$ vertices. Let $G$ be a maximal outerplanar graph with $n$ vertices. Since $G$ is 3-colourable, $G$ has an independent set $I$ of at least $n/3$ vertices. Let $S$ be the set of internal edges incident to any vertex in $I$. By Lemma 7, $G$ can be transformed by three simultaneous flips into a maximal outerplanar graph $G'$ containing no edge in $S$. Since $I$ is an independent set, each vertex in $I$ has degree two in $G'$. Let $G'' = G' \setminus I$.

Then $G''$ is maximal outerplanar with at most $2n/3$ vertices. By the inductive hypothesis, $G''$ can be transformed by $3(\log \frac{2n}{3}) / (\log 3 - 1)$ simultaneous flips into a maximal outerplanar graph $G'''$ whose dual has diameter at most $2(\log \frac{2n}{3}) / (\log 3 - 1)$. The total number of simultaneous flips is $3 + 3(\log \frac{2n}{3}) / (\log 3 - 1) = 3 \log n / (\log 3 - 1)$ as desired. Adding the vertices in $I$, which have degree two, back into $G'''$ increases the diameter of the dual tree by at most two, since degree two vertices correspond to leaves in the dual tree. Therefore, the diameter is $2 + 2(\log \frac{2n}{3}) / (\log 3 - 1) = 2 \log n / (\log 3 - 1)$ as desired. □

Lemmata 8 and 9 imply:

**Lemma 10.** *Let $v$ be a vertex of a maximal outerplanar graph $G$ with $n$ vertices. Then $G$ can be transformed by a sequence of at most $5 \log n / (\log 3 - 1) \approx 8.5 \log n$ simultaneous flips into a maximal outerplanar graph in which $v$ is adjacent to every other vertex.* □

### 3.2 Transforming one triangulation into another

We now describe how to transform a given $n$-vertex triangulation into any other $n$-vertex triangulation by a sequence of $O(\log n)$ simultaneous flips. As described at the start of Section 3, it suffices to prove that every triangulation $G$ can be transformed by $O(\log n)$ flips into the canonical triangulation. The first step is to transform $G$ into a Hamiltonian triangulation with one simultaneous flip by Corollary 1. Thus, our starting point is an $n$-vertex triangulation $G = (V, E)$ containing a Hamiltonian cycle $H$. The cycle $H$ naturally partitions $G$ into two subgraphs $G_I = (V, E_I)$ and $G_E = (V, E_O)$ such that $E_I \cap E_O = \emptyset$ and both $G_I \cup H$ and $G_O \cup H$ are maximal outerplanar.

At this point, it is tempting to apply Lemma 10 twice, once on $G_I \cup H$ and once on $G_O \cup H$ to reach the canonical triangulation. However, the lemma cannot be applied directly since we need to take into consideration the interaction between these two outerplanar subgraphs. The main problem is that an edge $e$ in $G_I$ may not be flippable since its corresponding flipped edge $e'$ may already be in $G_O$.

**Lemma 11.** *Suppose that an edge $e$ in $E_I$ can be flipped in $G_I \cup H$ to an edge $e'$, but $e$ cannot be flipped in $G$ because $e'$ is already in $G$. Then $e'$ can be flipped in $G$, provided that $n \geq 5$.*

*Proof.* The edge $e'$ must be in $G_O \cup H$. By Lemma 5, $e'$ can be flipped to an edge $e''$ in $G_O \cup H$. The only way that $e'$ is not flippable in $G$ is if $e''$ is in $G_I \cup H$. But as in Lemma 5, $e''$ cannot be in $G_I \cup H$. Therefore, $e'$ is flippable in $G$. □

**Lemma 12.** *A simultaneous flip $S$ in $G_I$ can be implemented by four simultaneous flips in $G$ (ignoring the effect in $G_O$).*

*Proof.* Let $S'$ be the set of edges in $S$ that are flippable in $G$. Let $S'' = S \setminus S'$. Let $S''_O$ be the edges in $G_O$ blocking the edges in $S''$. By Lemma 11, each edge in $S''_O$ is flippable. By Lemma 7, the edges in $S''_O$ can be removed with three simultaneous flips. Observe that $S'$ can be simultaneously flipped with one of the flips for $S''_O$. Once all of the edges in $S''_O$ are removed from $G$, the edges in $S''$ can be simultaneously flipped. In total we have four simultaneous flips. □

Lemmata 10 and 12 imply:

**Lemma 13.** *Let $v$ be a vertex of $G$. Then $G_I \cup H$ can be transformed by a sequence of at most $20 \log n / (\log 3 - 1)$ simultaneous flips into a maximal outerplanar graph in which $v$ is adjacent to every other vertex.* □

**Lemma 14.** *Let $G$ be a triangulation with $n$ vertices. Then $G$ can be transformed by a sequence of $25 \log n \,/\, (\log 3 - 1)$ simultaneous flips into the canonical triangulation.*

*Proof.* Apply one simultaneous flip to $G$ to obtain a Hamiltonian triangulation (Corollary 1). Define $H$, $G_I$ and $G_O$ as above. Let $vw$ be an edge of the Hamiltonian cycle. By Lemma 13, $G_I$ can be transformed by a sequence of at most $20 \log n \,/\, (\log 3 - 1)$ simultaneous flips into a maximal outerplanar graph in which $v$ is adjacent to every other vertex. By Lemma 10, $G_O$ can be transformed by a sequence of at most $5 \log n \,/\, (\log 3 - 1)$ simultaneous flips into a maximal outerplanar graph in which $w$ is adjacent to every other vertex. Observe that the edges in $G_I$ do not interfere with the flips in $G_O$ since every internal edge in $G_I$ is incident to $v$, and hence there are no edges incident to $v$ in $E_O$. The triangulation obtained is the canonical triangulation. The total number of flips is $1 + 25 \log n \,/\, (\log 3 - 1)$. □

*Proof of Theorem 3.* By Lemma 14, each of $G_1$ and $G_2$ can be transformed by $1+25 \log n \,/\, (\log 3-1)$ simultaneous flips into the canonical triangulation. To transform $G_1$ into $G_2$ first transform $G_1$ into the canonical transformation, and then apply the flips for $G_2$ in reverse order to transform the canonical triangulation into $G_2$. The total number of simultaneous flips is $2+50 \log n \,/\, (\log 3-1) \approx 87 \log n$. □

Note that although there are $O(\log n)$ simultaneous flips in Theorem 3, each of which may involve a linear number of edges, the total number of individual flips is $O(n)$ (see [2]). Thus our result also proves the result of Komuro [6] mentioned in Section 1 that $O(n)$ individual flips suffice to transform one $n$-vertex triangulation into any other.

## 4 Large Simultaneous Flips

In this section we prove bounds on the size of a maximum simultaneous flip in a triangulation. For any triangulation $G$, let $\mathsf{msf}(G)$ be the maximum size of a flippable set of edges in $G$. Note that Gao *et al.* [5] prove that every triangulation has at least $n-2$ (individually) flippable edges, and every triangulation with minimum degree four has at least $2n+3$ (individually) flippable edges.

**Lemma 15.** *For every triangulation $G$ with $n$ vertices, $\mathsf{msf}(G) \leq n-2$.*

*Proof.* Let $S$ be a flippable set of edges of $G$. Every edge in $S$ is incident to two distinct faces, and no other edge on each of these faces is in $S$. (Otherwise there would be two consecutive edges in $S$.) There are $2n-4$ faces in a triangulation. Thus $|S| \leq n-2$. □

**Theorem 4.** *For every triangulation $G = (V, E)$ with $n$ vertices, $\mathsf{msf}(G) \geq \frac{2}{3}(n-2)$.*

*Proof.* Let $V_4 = \{v \in V : \deg(v) = 4\}$. By $G[V_4]$ we denote the subgraph of $G$ induced by $V_4$. It is well known that the maximum degree of $G[V_4]$ is at most two, unless $G$ is the octahedron (that is, the 4-regular 6-vertex triangulation). The octahedron has a flippable set of $3 > \frac{2}{3}(n-2)$ edges. Henceforth assume that $G$ is not the octahedron. Thus each connected component of $G[V_4]$ is a cycle or a path. Let $C$ be a connected component of $G[V_4]$ that is a cycle. Then either $C$ is a face of $G$, in which case we say $C$ is a *facial component* of $G[V_4]$, or $G \setminus C$ consists of two vertices $v$ and $w$, each of which is adjacent to every vertex in $C$. In the latter case, as illustrated in Figure 4(a) and (b), $G$ has a flippable set of $n-3 \geq \frac{2}{3}(n-2)$ edges (since $n \geq |C \cup \{v,w\}| \geq 5$). We henceforth assume that every cycle in $G[V_4]$ is a facial component. Let $\mathcal{C}$ be the set of facial components of

$G[V_4]$. Let $P = (v_1, v_2, \ldots, v_k)$ be a connected component of $G[V_4]$ that is a $k$-vertex path. We call $P$ a *$k$-path component* of $G[V_4]$. It is easily seen that there exist two vertices in $G$, each of which is adjacent to every vertex in $P$, as illustrated in Figure 4(c). Let $\mathcal{P}_k$ be the set of $k$-vertex paths in $G[V_4]$.

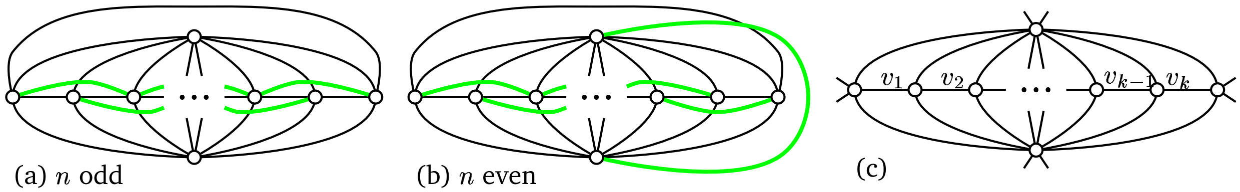


Figure 4: (a) & (b) The triangulation with a non-facial cycle of degree four vertices has a flippable set of $n-3$ edges. (c) A $k$-path component.

For every edge $vw$ of $G$, let $pvw$ and $qvw$ be the faces incident to $vw$. We say that the vertices $p$ and $q$ *see* $vw$. Edges $vw$ and $xy$ are a *bad pair* if $vw$ and $xy$ are seen by the same pair of vertices. An edge is *bad* if it is a member of a bad pair. An edge is *good* if it is not bad. Observe that every edge incident to a vertex of degree four is bad.

Construct an auxiliary graph $G'$ from $G$ as follows. First, delete each facial component of $G[V_4]$. (Note that this operation and those to follow are *not* reiterated on the produced graph.) Now consider a $k$-path component $P = (v_1, v_2, \ldots, v_k)$ in $G[V_4]$ for some $k \geq 1$. Let the neighbours of $P$ be the 4-cycle $(a, b, c, d)$ such that $bv_i$ and $dv_i$ are edges of $G$ for all $i$, $1 \leq i \leq k$. We say $P$ is *$ac$-connected* (respectively, *$bd$-connected*) if $ac$ ($bd$) is an edge of $G$, or there exists an edge $e$ not incident to a vertex in $P$ such that $e$ is seen by $a$ and $c$ ($b$ and $d$). Note that $P$ is not both $ac$-connected and $bd$-connected, as otherwise there would be a subdivision of $K_5$. If $P$ is $ac$-connected then in $G'$, delete $P$ and merge $b$ and $d$ into a single vertex $x_v$, as illustrated in Figure 5(a). If $P$ is $bd$-connected then in $G'$, delete $P$ and merge $a$ and $c$ into a single vertex $x_v$, as illustrated in Figure 5(b). In the case that $P$ is neither $ac$-connected or $bd$-connected, apply either reduction.

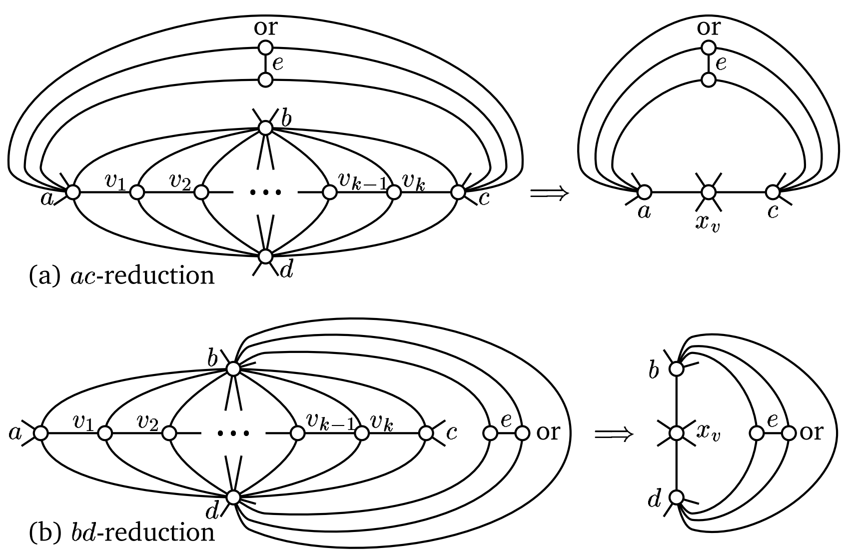


Figure 5: Reducing a path component.

In each merge operation, replace parallel edges on a single face of $G'$ by a single edge. Thus $G'$ is a plane multigraph with each face a triangle. Note that good edges of $G$ are preserved in $G'$. Let $n'$ be the number of vertices of $G'$. Then

$$n' = n - 3|\mathcal{C}| - \sum_{k\geq 1}(k+1)|\mathcal{P}_k| \ . \tag{1}$$

Let $\{E_1', E_2', E_3'\}$ be the colour classes of a Tait edge 3-colouring of $G'$ (see Lemma 4). For each $i \in \{1,2,3\}$, let $S_i'$ be the subset of $E_i'$ consisting of those edges corresponding to good edges in $G$.

If two bad edges are consecutive in $G$ then their common endpoint has degree four. Since all the degree four vertices of $G$ are not in $G'$, no two bad edges of $G$ appear on a single face of $G'$. Thus the number of bad edges of $G$ also in $G'$ is at most $n'-2$. Thus the number of good edges of $G$ also in $G'$ is at least $2(n'-2)$. By (1) we have

$$|S_1'| + |S_2'| + |S_3'| \geq 2(n'-2) \geq 2(n-2) - 6|\mathcal{C}| - 2\sum_{k\geq 1}(k+1)|\mathcal{P}_k| \ . \tag{2}$$

We claim that each $S_i'$ is a flippable set of edges in $G$. Suppose that this is not the case. That is, there exist parallel edges $e$ and $f$ in $G\langle S_i'\rangle$. It is not the case that both $e$ and $f$ are edges of $G$, as otherwise they would form a bad pair, and all edges in $S_i'$ are good. It is not the case that both $e$ and $f$ are not edges of $G$, as otherwise they would have been flipped from a bad pair of edges of $G$ in $S_i'$, and all edges in $S_i'$ are good. The remaining case is that one of $e$ and $f$ is in $G$ and the other is not in $G$. Say $e = vw$ is in $G$. Let $pq \in S_i'$ be the edge of $G$ that was flipped into $f$. Observe that $pv$ is consecutive with $pq$, $pw$ is consecutive with $pq$, and $vwp$ is a triangle. Thus $vw$ and $pq$ receive the same colour in the Tait colouring. As proved in Lemma 1, $vw$ is good. Thus $vw \in S_i'$, and $vw$ is not in $G\langle S_i\rangle$, which is the desired contradiction. Thus each set $S_i'$ is flippable in $G$.

Fix $i \in \{1,2,3\}$, and initialise $S_i$ to be the edges of $G$ corresponding to the set $S_i'$. For each component of $G[V_4]$, we now add edges to $S_i$ so that it remains a flippable set of edges. Consider a vertex $v$ of degree $4$ in $G$ that is isolated in $G[V_4]$. Let $(p,q,r,s)$ be the neighbours of $v$ in $G$ in cyclic order defined by the embedding, such that $q$ and $s$ are merged into a vertex $x$ in $G'$. For all $i,j \in \{1,2,3\}$, let $A^{ij}$ be the set of isolated vertices in $G[V_4]$ such that the corresponding edges $xp \in E_i$ and $xr \in E_j$ (or $xr \in E_i'$ and $xp \in E_j'$). By the choice of reduction rule for $v$, $qs$ is not an edge of $G$. For each $v \in A^{ii}$, replace $xp$ and $xr$ in $S_i'$ by $\{pq, qr, rs, sp\}$ in $S_i$. As illustrated in Figure 6(a), these four edges are simultaneously flippable in $G$, since the degrees of $p$ and $r$ are both at least 5. For each $v \in A^{ij}$ with $j \neq i$, replace $xp$ in $S_i'$ by $\{pq, ps, qs\}$ in $S_i$. Since $qs$ is not an edge of $G$, these three edges are simultaneously flippable in $G$, as illustrated in Figure 6(b). For each $v \in A^{j\ell}$ with $j \neq i$ and $\ell \neq i$, add $qs$ to $S_i$. Since $qs$ is not an edge of $G$, $qs$ is flippable in $G$, as illustrated in Figure 6(c). Consider a face-component $\{u,v,w\}$ of $G[V_4]$, and let $\{p,q,r\}$ be the vertices adjacent to $u$, $v$, and $w$, such that $up, vq, wr \notin E$. Then $pqr$ is a separating triangle of $G$, and thus each of the edges $\{pq, pr, qr\}$ are good. Every triangle of $G$ receives three distinct colours in the Tait edge 3-colouring. Thus exactly one of $\{pq, pr, qr\}$ is in $S_i'$. Suppose without loss of generality that $pq \in S_i'$. Then $\{vq, wr\}$ added to $S_i$ forms a flippable set of edges in $G$, as illustrated in Figure 6(d).

Let $B_{ij}$ be the set of $k$-vertex path components $P$ of $G[V_4]$ with $k \geq 2$, such that if $P$ is replaced by the 2-edge path $pxr$ in $G'$ then $xp \in E_i'$ and $xr \in E_j'$ (or $xr \in E_i'$ and $xp \in E_j'$). For every $k$-path component of $G[V_4]$ with $k \geq 2$, $k$ edges can be added to $S_i$ (see [2] for the proof, and Figure 4(a) and (b) for the basic idea). Let $\{j, \ell\} = \{1,2,3\} \setminus \{i\}$. Then

$$|S_i| = |S_i'| + 2|\mathcal{C}| + 2|A^{ii}| + 2|A^{ij}| + 2|A^{i\ell}| + |A^{j\ell}| + \sum_{k\geq 2} k \cdot |\mathcal{P}_k| \ .$$

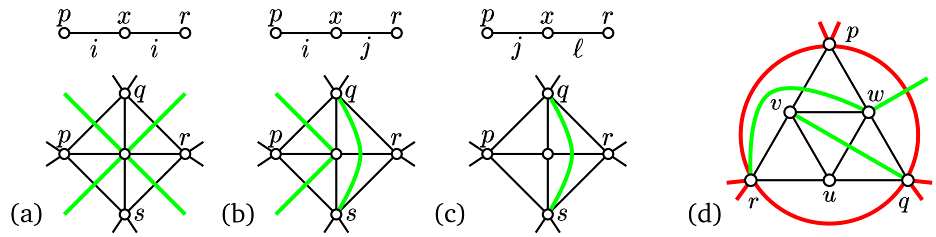


Figure 6: Adding to $S_i$: (a)-(c) for an isolated degree 4 vertex, and (d) for a facial component.

Thus,

$$
\begin{aligned}
& |S_1| + |S_2| + |S_3| \\
&= |S'_1| + |S'_2| + |S'_3| + 6|\mathcal{C}| + 4(|A^{11}| + |A^{22}| + |A^{33}|) + 5(|A^{12}| + |A^{13}| + |A^{23}|) + 3\sum_{k\geq 2} k \cdot |\mathcal{P}_k| \\
&\geq |S'_1| + |S'_2| + |S'_3| + 6|\mathcal{C}| + 4|P_1| + 3\sum_{k\geq 2} k \cdot |\mathcal{P}_k| \ .
\end{aligned}
$$

By (2),

$$
\begin{aligned}
|S_1| + |S_2| + |S_3| \geq \ & 2(n-2) - 6|\mathcal{C}| - 2\sum_{k\geq 1}(k+1)|\mathcal{P}_k| + 6|\mathcal{C}| + 4|P_1| + 3\sum_{k\geq 2} k \cdot |\mathcal{P}_k| \\
\geq \ & 2(n-2) + \sum_{k\geq 2}(k-2)|\mathcal{P}_k| \quad \geq \ 2(n-2) \ .
\end{aligned}
$$

Thus one of $\{S_1, S_2, S_3\}$ is a flippable set of at least $\frac{2}{3}(n-2)$ edges in $G$. □

**Lemma 16.** *There exists an infinite family $\mathcal{F}$ of triangulations such that $\mathsf{msf}(G) = \frac{6}{7}(n-2)$ for every $n$-vertex triangulation $G \in \mathcal{F}$.*

*Proof.* Let $G_0$ be an arbitrary triangulation with $n_0$ vertices. Let $G$ be the triangulation obtained from $G_0$ by adding a triangle of three vertices inside each face $uvw$ of $G$, each of which is adjacent to two of $\{u, v, w\}$. Let $n$ be the number of vertices of $G$. Then $n-2 = n_0+3(2n_0-4)-2 = 7(n_0-2)$. Let $S$ be a flippable set of edges of $G$. For every face of $G_0$, at least one of the corresponding seven faces of $G$ does not have an edge in $S$, as illustrated in Figure 7(a). Thus at least $2(n_0-2) = \frac{2}{7}(n-2)$ faces of $G$ do not have an edge in $S$. Every face of $G$ has at most one edge in $S$. Thus $|S| \leq \frac{1}{2}(2(n-2) - \frac{2}{7}(n-2)) = \frac{6}{7}(n-2)$. It remains to construct a flippable set of $\frac{6}{7}(n-2)$ edges in $G$. For each face of $G_0$, add the edges shown in Figure 7(b) to a set $S$. Clearly $S$ is flippable. In every face of $G_0$, exactly one of the corresponding seven faces of $G$ does not have an edge in $S$, and the remaining six faces each have exactly one edge in $S$. By the above analysis, $|S| = \frac{6}{7}(n-2)$. □

An obvious open problem is to close the gap between the lower bound of $\frac{2}{3}(n-2)$ and the upper bound of $\frac{6}{7}(n-2)$ in the above results. Note that every 5-connected triangulation $G$ has $\mathsf{msf}(G) = n-2$ (see [2]).

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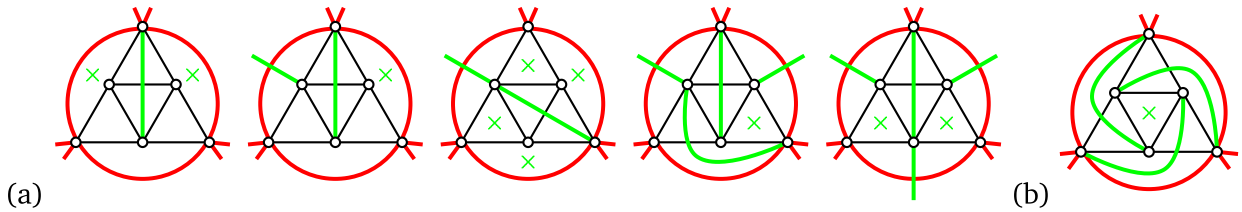

Figure 7: (a) For any number of flips in the outer triangle, at least one internal face does not have an edge in $S$. (b) How to construct a flip set for $G$.

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