COMP 3804 — Tutorial February 9

Problem 1: You are given three beer barrels B_1, B_2, and B_3. Barrel B_1 has a capacity of 8 litres, barrel B_2 has a capacity of 5 litres, and barrel B_3 has a capacity of 3 litres.

At any moment, each barrel contains a given amount of beer (in litres). In one *step*, you can pour beer from one barrel, say B_i, to another barrel, say B_j. This step terminates at the moment when B_i becomes empty or B_j becomes full, whichever happens first.

To give some examples:

- If B_1 contains 6 litres of beer, B_2 contains 2 litres of beer, and B_3 contains 0 litres of beer, then we can pour the entire contents of barrel B_2 to barrel B_3. At the end of this step, B_1 contains 6 litres of beer, B_2 contains 0 litres of beer, and B_3 contains 2 litres of beer.

- If B_1 contains 3 litres of beer, B_2 contains 4 litres of beer, and B_3 contains 1 litre of beer, then we can pour 2 litres of beer from B_1 to B_3. At the end of this step, B_1 contains 1 litre of beer, B_2 contains 4 litres of beer, and B_3 contains 3 litres of beer.

Decision problem:

- Let b_1, b_2, and b_3 be integers such that $b_1 \geq 0$, $b_2 \geq 0$, $0 \leq b_3 \leq 3$, and $b_1 + b_2 + b_3 = 4$. Similarly, let b'_1, b'_2, and b'_3 be integers such that $b'_1 \geq 0$, $b'_2 \geq 0$, $0 \leq b'_3 \leq 3$, and $b'_1 + b'_2 + b'_3 = 4$.

- Initially, barrel B_1 is filled with b_1 litres of beer, barrel B_2 is filled with b_2 litres of beer, and barrel B_3 is filled with b_3 litres of beer.

- We want to decide whether or not it is possible to perform a sequence of steps that results in barrel B_1 having b'_1 liters of beer, barrel B_2 having b'_2 litres of beer, and barrel B_3 having b'_3 litres of beer?

(1.1) Formulate this as a problem on a directed graph. What are the vertices of the graph? What are the directed edges of the graph?

(1.2) Draw the entire graph.

(1.3) Assume that $(b_1, b_2, b_3) = (4, 0, 0)$ and $(b'_1, b'_2, b'_3) = (3, 1, 0)$. Use your graph to decide whether the answer to the decision problem is YES or NO.

(1.4) Assume that $(b_1, b_2, b_3) = (4, 0, 0)$ and $(b'_1, b'_2, b'_3) = (2, 1, 1)$. Use your graph to decide whether the answer to the decision problem is YES or NO.

Problem 2: Let $G = (V, E)$ be an undirected graph. A *vertex coloring* of G is a function $f : V \rightarrow \{1, 2, \ldots, k\}$ such that for every edge $\{u, v\}$ in E, $f(u) \neq f(v)$. In words, each vertex u gets a "color" $f(u)$, from a set of k "colors", such that the two vertices of each edge have different colors.

Assume that the graph G has exactly one cycle with an odd number of vertices. (The graph may contain cycles with an even number of vertices.)

What is the smallest integer k such that a vertex coloring with k colors exists? As always, justify your answer.

Problem 3: Let $G = (V, E)$ be a directed graph, which is given to you in the adjacency list format. Thus, each vertex u has a list that stores all vertices of the set

$$\{v : (u, v) \in E\}.$$

The backwards graph G_b is obtained from G by replacing each edge (u, v) in G by the edge (v, u). In words, in G_b, we follow the edges of G backwards.

Describe an algorithm that computes, in $O(|V| + |E|)$ time, an adjacency list representation of G_b. As always, justify your answer and the running time of your algorithm.