COMP 2804 — Assignment 4

Due: Sunday December 6, 11:55 pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm", or "my dog ate my laptop charger".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1:

- Write your name and student number.

Question 2: Consider six fair dice $D_1, \ldots, D_6$, each one having six faces. For each i with $1 \leq i \leq 6$, the die D_i has one face labeled i, whereas its other five faces are labeled zero.

You roll each of these dice once. Consider the random variable X, whose value is the sum of the results of these six rolls.

Determine the expected value $\mathbb{E}(X)$ of X.

Question 3: Consider a standard red die and a standard blue die; both of them are fair. You roll each die once. Consider the random variables

$$\begin{aligned} X &= \text{the result of the red die plus the result of the blue die,} \\ Y &= \text{the result of the red die minus the result of the blue die.} \end{aligned}$$

1. Prove that $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$.
2. Are X and Y independent random variables? As always, justify your answer.

Question 4: When FX and his girlfriend XF have a child, this child is a boy with probability 1/2 and a girl with probability 1/2, independently of the sex of their other children. FX and XF stop having children as soon as they have a girl or four children.

Consider the random variables

$$\begin{aligned} C &= \text{the number of children that FX and XF have,} \\ B &= \text{the number of boys that FX and XF have.} \end{aligned}$$

Determine the expected values $\mathbb{E}(C)$ and $\mathbb{E}(B)$.

Question 5: Consider the following algorithm, which takes as input an integer $n \geq 1$:

Algorithm MYSTERY(n)**:**

> // all random choices made are mutually independent
> $X = 0$;
> **for** $i = 1$ **to** n
> **do** a = random real number between -1 and 1;
> b = random real number between -1 and 1;
> **if** $a^2 + b^2 \leq 1$
> **then** $X = X + 1$
> **endif**
> **endfor**;
> $Y = \frac{4}{n} \cdot X$;
> return Y

The output Y of this algorithm is a random variable. Determine the expected value $\mathbb{E}(Y)$ of this random variable.

Hint: If you implement this algorithm and run it several times for large values of n, then you may recognize the output. For the derivation of your value of $\mathbb{E}(Y)$, use indicator random variables.

Question 6: In the Lotto 6/49 lottery, you pick a 6-element subset X from the set $N = \{1, ..., 49\}$ and then a machine picks, uniformly at random, a 6-element subset Y from N.

1. The number $|X \cap Y|$ of numbers you picked correctly is a random variable. What is the expected value of this random variable? Give an exact answer, some Python code is provided below that can help you with the calculation.

2. [Warning: The following is an oversimplification, don't use it to make life choices.] The payout in Lotto 6/49 is relative to the Jackpot, which we will call x. The payout is defined as follows:

- 6 correct numbers: x
- 5 correct numbers $x/95$
- 4 correct numbers $x/4365$
- 3 correct numbers 10
- 2 correct numbers 3

If you are given one Lotto 6/49 ticket, what is your expected payout? Give an exact answer (which will include the variable x).

3. A Lotto 6/49 ticket costs \$3. What is the minimum jackpot value x that gives a payout of at least \$3?

Useful Python Code

```
morin@lauteschwein:~$ python3
Python 3.8.5 (default, Jul 28 2020, 12:59:40)
[GCC 9.3.0] on linux
Type "help", "copyright", "credits" or "license" for more information.
>>> from fractions import Fraction
>>> from math import factorial
>>> binom=lambda n,k: Fraction(factorial(n),factorial(k)*factorial(n-k))
>>> n = 5
>>> print(binom(n,3)/2**n)
5/16
>>> sum([binom(n,k) for k in range(n+1)])
Fraction(32, 1)
>>> sum([binom(n,k)/2**n for k in range(n+1)])
Fraction(1, 1)
```

Question 7: Consider the following algorithm, which takes as input an integer $n \geq 1$:

Algorithm Euler(n)**:**

> // all random choices made are mutually independent
> $total = 0$;
> $i = 0$;
> **while** $total \leq n$
> **do** $i = i + 1$;
> x_i = uniformly random element in $\{1, 2, \ldots, n\}$;
> $total = total + x_i$
> **endwhile**;
> return i

The output i of this algorithm is a random variable, which we denote by X.

1. What are the possible values that X can take? As always, justify your answer.

2. Let k be an integer with $0 \leq k \leq n$. Prove that

$$\Pr(X \geq k+1) = \binom{n}{k} \cdot (1/n)^k.$$

   Hint: What is the number of solutions of the inequality $x_1 + x_2 + \cdots + x_k \leq n$, where $x_1, x_2, \ldots, x_k$ are strictly positive integers?

3. Determine $\mathbb{E}(X)$.

   Hint: You may use the fact that $\mathbb{E}(X) = \sum_{k=0}^{n} \Pr(X \geq k+1)$.

4. Determine $\lim_{n\to\infty} \mathbb{E}(X)$.

Question 8: Let G be an undirected graph with n vertices and m edges. This graph is not necessarily connected. Recall that the degree of a vertex u, denoted $deg(u)$, is the number of edges that are incident on u. We assume that every vertex has degree at least one. If u and v are two vertices that are connected by an edge, then we say that v is a neighbor of u.

Consider the following experiment:

- Let x be a uniformly random vertex.
- Let y be a uniformly random neighbor of x.
- Let $X = deg(x)$ and $Y = deg(y)$.

1. Let $a > 0$ and $b > 0$ be real numbers. Prove that

$$\frac{a}{b} + \frac{b}{a} \geq 2,$$

   with equality if and only if $a = b$.

   Hint: Rewrite this inequality until you get an equivalent inequality which obviously holds.

2. Prove that the expected value $\mathbb{E}(X)$ of the random variable X satisfies

$$\mathbb{E}(X) = \frac{2m}{n}.$$

3. Prove that the expected value $\mathbb{E}(Y)$ of the random variable Y satisfies

$$\mathbb{E}(Y) = \frac{1}{n} \cdot \sum_{u:\text{ vertex in } G} \left(\sum_{v:\text{ neighbor of } u} \frac{deg(v)}{deg(u)} \right).$$

4. Prove that

$$\sum_{u\text{: vertex in } G} \left(\sum_{v\text{: neighbor of } u} \frac{deg(v)}{deg(u)} \right) = \sum_{\{u,v\}\text{: edge in } G} \left(\frac{deg(v)}{deg(u)} + \frac{deg(u)}{deg(v)} \right).$$

5. Prove that

$$\mathbb{E}(Y) \geq \mathbb{E}(X),$$

with equality if and only if each connected component of G is regular (i.e., all vertices in the same connected component have the same degree).