COMP 2804 — Assignment 2

Due: Sunday October 11, 11:55 pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm", or "my dog ate my laptop charger".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1:

- Write your name and student number.

Question 2: Consider a finite set S of points in the two-dimensional plane. A point p of S is called a *covidiot*, if p is within two meters of some other point of S.

Let R be a rectangle whose horizontal sides have a length of 20 meters and whose vertical sides have a length of 30 meters. Assume that all points of S are contained in R and that S contains at least 601 points.

- Use the Pigeonhole Principle to prove that S contains at least two covidiots.

Question 3:

- Let $\alpha_1, \alpha_2, \ldots, \alpha_7$ be real numbers that are all contained in the interval $[-\pi/2, \pi/2]$.

  Use the Pigeonhole Principle to prove that there are two distinct indices i and j such that $0 \leq \alpha_i - \alpha_j \leq \pi/6$.

- Let $a_1, a_2, \ldots, a_7$ be real numbers such that $a_i a_j \neq -1$ for all $i \neq j$.

  Prove that there are two distinct indices i and j such that

  $$0 \leq \frac{a_i - a_j}{1 + a_i a_j} \leq \frac{1}{\sqrt{3}}.$$

  Hint: For each i, let p_i be the point with coordinates $(1, a_i)$, and consider the angle between the x-axis and the vector from the origin to p_i. You learned in high school that

  $$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

Question 4: The function $f : \{0, 1, 2, \ldots\} \to \mathbb{R}$ is defined by

$$\begin{array}{rcl} f(0) & = & 7, \\ f(n) & = & 5 \cdot f(n-1) + 12n^2 - 30n + 15 \quad \text{if } n \geq 1. \end{array}$$

- Prove that for every integer $n \geq 0$,

  $$f(n) = 7 \cdot 5^n - 3n^2.$$

Question 5: Consider the following recursive algorithm, which takes as input an integer $n \geq 1$ that is a power of two:

Algorithm $\text{MYSTERY}(n)$**:**

> **if** $n = 1$
> **then** return 1
> **else** $x = \text{MYSTERY}(n/2)$;
> return $n + x$
> **endif**

- Determine the output of algorithm $\text{MYSTERY}(n)$ as a function of n. As always, justify your answer.

Question 6: In class, we have seen that for any integer $m \geq 1$, the number of 00-free bitstrings of length m is equal to f_{m+2}, which is the $(m+2)$-th Fibonacci number.

Let $n \geq 4$ be an integer and consider the set

$$B = \{(x, y) : \text{both } x \text{ and } y \text{ are 00-free bitstrings of length } n - 1\}.$$

- Explain, in a few sentences, why $|B| = f_{n+1}^2$.

  Note that f_{n+1}^2 is the square of f_{n+1}, i.e., it is to be read as $(f_{n+1})^2$.

- Determine the number of elements (x, y) in B for which the concatenation xy is 00-free.

- Determine the number of elements (x, y) in B for which the concatenation xy is not 00-free.

- Use the previous results to prove that

$$f_{2n} = f_{n+1}^2 - f_{n-1}^2.$$

Question 7: The sequence $s_0, s_1, s_2, \ldots$ of bitstrings is recursively defined as follows:

1. $s_0 = 1$, i.e., the bitstring of length one consisting of one 1.

2. For $n \geq 1$, the bitstring s_n is obtained as follows: In the bitstring s_{n-1}, replace each 1 by 10 and replace each 0 by 1.

The first few strings in this sequence are

$$s_0 = 1; s_1 = 10; s_2 = 101; s_3 = 10110; s_4 = 10110101.$$

- Prove that for each $n \geq 0$, the bitstring s_n is 00-free.

For each $n \geq 0$, let L_n be the length of s_n and let O_n be the number of 1's in s_n.

- Determine L_0, L_1, O_0, and O_1.

- Let $n \geq 2$. Prove that $O_n = L_{n-1}$.

- Let $n \geq 2$. Prove that $L_n = L_{n-1} + O_{n-1}$.

- Express O_n in terms of numbers that we have seen in class.

For each $n \geq 0$, let Z_n be the number of 0's in s_n.

- Express Z_n in terms of numbers that we have seen in class.

Question 8: Consider a bitstring $b_1 b_2 \ldots b_n$ of length n, where $n \geq 2$ is an integer. For any integer i, the bit b_i is called *lonely*, if $b_i = 1$ and its neighboring bits are 0. More formally, the bit b_i is lonely if

1. $i = 1$, $b_1 = 1$, and $b_2 = 0$, or

2. $i = n$, $b_n = 1$, and $b_{n-1} = 0$, or

3. $2 \leq i \leq n-1$, $b_i = 1$, and $b_{i-1} = b_{i+1} = 0$.

For example, in the following bitstring, the three bits in boldface are lonely:

$$\mathbf{1}00011100\mathbf{1}0\mathbf{1}000$$

A bitstring of length at least two is called *happy* if none of its bits is lonely.

For any integer $n \geq 2$, let A_n denote the number of happy bitstrings of length n, and let B_n denote the number of happy bitstrings of length n that start with 11.

- Determine A_2, A_3, A_4, and A_5.
- Let $n \geq 3$ be an integer. Express A_n in terms of A_{n-1} and B_n.
- Let $n \geq 4$ be an integer. Express A_{n-1} in terms of A_{n-2} and B_{n-1}.
- Let $n \geq 5$ be an integer. Express B_n in terms of A_{n-3} and B_{n-1}.
- Let $n \geq 5$ be an integer. Prove that

$$A_n = 2 \cdot A_{n-1} - A_{n-2} + A_{n-3}.$$