Region-Fault Tolerant Geometric Spanners

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Abstract

We introduce the concept of region-fault tolerant spanners for planar point sets, and prove the existence of region-fault tolerant spanners of small size. For a geometric graph $\mathcal{G}$ on a point set P and a region F, we define $\mathcal{G}\ominus F$ to be what remains of $\mathcal{G}$ after the vertices and edges of $\mathcal{G}$ intersecting F have been removed. A *$\mathcal{C}$-fault tolerant t-spanner* is a geometric graph $\mathcal{G}$ on P such that for any convex region F, the graph $\mathcal{G}\ominus F$ is a t-spanner for $\mathcal{G}_c(P)\ominus F$, where $\mathcal{G}_c(P)$ is the complete geometric graph on P. We prove that any set P of n points admits a $\mathcal{C}$-fault tolerant $(1+\varepsilon)$-spanner of size $\mathcal{O}(n \log n)$, for any constant $\varepsilon > 0$; if adding Steiner points is allowed then the size of the spanner reduces to $\mathcal{O}(n)$, and for several special cases we show how to obtain region-fault tolerant spanners of $\mathcal{O}(n)$ size without using Steiner points. We also consider *fault-tolerant geodesic t-spanners*: this is a variant where, for any disk D, the distance in $\mathcal{G}\ominus D$ between any two points $u, v \in P \setminus D$ is at most t times the geodesic distance between u and v in $\mathbb{R}^2 \setminus D$. We prove that for any P we can add $\mathcal{O}(n)$ Steiner points to obtain a fault-tolerant geodesic $(1+\varepsilon)$-spanner of size $\mathcal{O}(n)$.

1 Introduction

A *geometric network* on a set P of points in d-dimensional space is an undirected graph $\mathcal{G}(P, E)$ with vertex set P whose edges are straight-line segments connecting pairs of points in P. Often the space considered is the Euclidean plane—this is also the setting we shall consider—but other metrics and/or higher dimensions can be considered as well. Geometric networks naturally model many real-life networks, such as road networks, telecommunication networks, and so on.

When designing a network for a given set P of points, several criteria can be taken into account. In particular, in many applications it is important to ensure a fast connection between every pair of points in P. For this it would be ideal to have a direct connection between every pair of points—the network would then be a complete graph—but in most applications this is unacceptable due to the high costs. This leads to the concepts of spanners, as defined below. Spanners were introduced by Peleg and Schäffer [15] in the context of distributed computing and by Chew [3] in a geometric context.

For two vertices u, v in a weighted graph $\mathcal{G}$ we use $d_{\mathcal{G}}(u, v)$ to denote their distance in the graph, that is, the length of the (weighted) shortest path between them. Now consider a weighted graph $\mathcal{G}(V, E)$ and a graph $\mathcal{G}'(V, E')$ on the same vertex set but with edge set $E' \subseteq E$. We say that $\mathcal{G}'$ is a *t-spanner* of $\mathcal{G}$ if for each pair of vertices $u, v \in V$ we have that $d_{\mathcal{G}'}(u, v) \leq t \cdot d_{\mathcal{G}}(u, v)$. The *dilation* or *stretch factor* of $\mathcal{G}'$ is the minimum t for which $\mathcal{G}'$ is a t-spanner of $\mathcal{G}$.

For geometric networks the weight of an edge (u, v) is defined to be the Euclidean distance $d(u, v)$ between u and v. Now we say that a geometric network $\mathcal{G}(P, E)$ is a *(geometric) t-spanner* if $\mathcal{G}(P, E)$ is a t-spanner of $\mathcal{G}_c(P)$, where $\mathcal{G}_c(P)$ is the complete geometric network on P. In other words, for any two points $p, q \in P$ the graph distance in $\mathcal{G}$ is at most t times the Euclidean distance between the two points. Geometric spanners have received a lot of attention over the past few years—see the survey papers [8, 10, 19] and the book by Narasimhan and Smid [14]. From now on, we shall limit our discussion to geometric spanners.

The spanner concept captures the notion of "good" networks when short connections between the points are important. The main question is whether spanners exist that have a small stretch factor and a small, ideally near-linear, number of edges. Other desirable properties of a spanner are for example that the total weight of the edges is small, or that the maximum degree is low. As it turns out, such spanners do indeed exist: it has been shown that for any set P of n points and for any fixed $\varepsilon > 0$ there exists a $(1+\varepsilon)$-spanner with $\mathcal{O}(n)$ edges, bounded degree, and whose total weight is $\mathcal{O}(wt(MST(P)))$, where $wt(MST(P))$ is the weight of a minimum spanning tree of P [6].

Another useful property of a network is fault tol-

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erance: after one or more vertices or edges fail, the spanner should retain its good properties. In particular, there should still be a short path between any two vertices in what remains of the spanner after the fault. Levcopolous et al. [12] showed the existence of k-vertex (or: k-edge) fault tolerant geometric spanners with $\mathcal{O}(nk \log n)$ edges. This was improved by Lukovszki [13], who presented a fault-tolerant spanner with $\mathcal{O}(nk)$ edges, which is optimal. Later Czumaj and Zhao [5] showed that a greedy approach produces a k-vertex (or: k-edge) fault tolerant geometric $(1+\varepsilon)$-spanner with degree $\mathcal{O}(k)$ and total weight $\mathcal{O}(k^2 \cdot wt(MST(P)))$; these bounds are asymptotically optimal.

The papers on fault-tolerant spanners mentioned above all consider faults that can destroy an arbitrary collection of k vertices or edges. For geometric spanners, however, it is natural to consider *region faults*: faults that do not destroy an arbitrary collection of vertices and edges, but faults that destroy all vertices and edges intersecting some geometric fault region. This is relevant, for instance, when the spanner models a road network and a natural (or other) disaster makes all the roads in some region inaccessible. This is the topic of our paper: we study the existence of small spanners in the plane[1] that are tolerant against region faults. Before we present our results, let us define region-fault tolerance more precisely.

Let $\mathcal{F}$ be a family of regions in the plane, which we call the *fault regions*. For a fault region $F \in \mathcal{F}$ and a geometric graph $\mathcal{G}$ on a point set P, we define $\mathcal{G} \ominus F$ to be the part of $\mathcal{G}$ that remains after the points from P inside F and all edges that intersect F have been removed from the graph—see Fig. 1. (For simplicity we assume that a region fault F does not contain its boundary so that vertices and edges that do not intersect the interior of F will not be affected.)

An *$\mathcal{F}$-fault tolerant t-spanner* is a geometric graph $\mathcal{G}$ on P such that for any region $F \in \mathcal{F}$, the graph $\mathcal{G} \ominus F$ is a t-spanner for $\mathcal{G}_c(P) \ominus F$. (Recall that $\mathcal{G}_c(P)$ is the complete geometric graph on P.) We are mainly interested in the case where $\mathcal{F}$ is the family $\mathcal{C}$ of convex sets.[2]

We shall also consider the case where we are allowed to add Steiner points to the graph. In other words, instead of constructing a geometric network for P, we are allowed to construct a network for $P \cup Q$ for some set Q of Steiner points. In this case, we only require short connections between the points in P. Thus we say that a graph $\mathcal{G}$ on $P \cup Q$ is a *$\mathcal{C}$-fault tolerant Steiner t-spanner* for P if, for any $F \in \mathcal{C}$ and any two points $u, v \in P \setminus F$, the distance between u and v in $\mathcal{G} \ominus F$ is at most t times their distance in $\mathcal{G}_c(P) \ominus F$.

We also study another variant of region-fault tolerance. In this variant we require that the distance between any two points u, v in $\mathcal{G} \ominus F$ is at most t times the geodesic distance between u and v in $\mathbb{R}^2 \setminus F$. Note that the geodesic distance in $\mathbb{R}^2 \setminus F$—that is, the length of a shortest path in $\mathbb{R}^2 \setminus F$—is never more than the distance between u and v in $\mathcal{G}_c(P) \ominus F$. We call a spanner with this property an *$\mathcal{F}$-fault tolerant geodesic spanner*. It is not difficult to show that finite size $\mathcal{F}$-fault tolerant geodesic spanners do not exist unless we are allowed to use Steiner points. Even in the case of Steiner points, finite size $\mathcal{F}$-fault tolerant geodesic spanners do not exist when $\mathcal{F}$ is the family $\mathcal{C}$ of all convex sets. Hence, we restrict our attention to $\mathcal{D}$-fault tolerant geodesic spanners, where $\mathcal{D}$ is the family of disks in the plane.

We obtain the following results.

- In Section 2 we present a general method to convert a well-separated pairs decomposition (WSPD) [1] for P into a $\mathcal{C}$-fault tolerant spanner for P. We use this method to obtain linear-size $\mathcal{C}$-fault tolerant $(1+\varepsilon)$-spanners for points in convex position and for points distributed uniformly at random inside the unit square, and to obtain linear-size $\mathcal{C}$-fault tolerant Steiner $(1+\varepsilon)$-spanners for arbitrary point sets.

- In Section 3 we study small $\mathcal{C}$-fault tolerant (non-Steiner) spanners for arbitrary point sets. By combining a more relaxed version of the WSPD with ideas from Θ-graphs [11], we show that any point set P admits a $\mathcal{C}$-fault tolerant $(1+\varepsilon)$-spanner of size $\mathcal{O}(n \log n)$.

- In Section 4 we study the geodesic case. We show that for any set P of n points there exists a $\mathcal{D}$-fault tolerant geodesic Steiner $(1+\varepsilon)$-spanner with $\mathcal{O}(n/\varepsilon^6)$ edges and $\mathcal{O}(n/\varepsilon^5)$ Steiner points.

2 Constructing $\mathcal{C}$-fault tolerant spanners using the WSPD

In this section we show a general method to obtain a $\mathcal{C}$-fault tolerant spanner from a well-separated-pair decomposition of a point set P. Before we start we prove a general lemma showing that, when constructing $\mathcal{C}$-fault tolerant spanners, we can in fact restrict our

[1] The concepts and many of the results carry over to d-dimensional Euclidean space. However, we feel the concept is mainly interesting in the plane, so we confine ourselves to the planar case in this extended abstract.

[2] It is easy to see that there are no small region-fault tolerant t-spanners with respect to non-convex faults: if $\mathcal{HH}$ denotes the family of regions that are the union of two half-planes, then $\mathcal{G}_c(P)$ is the only $\mathcal{HH}$-fault tolerant t-spanner for P, for any finite t.

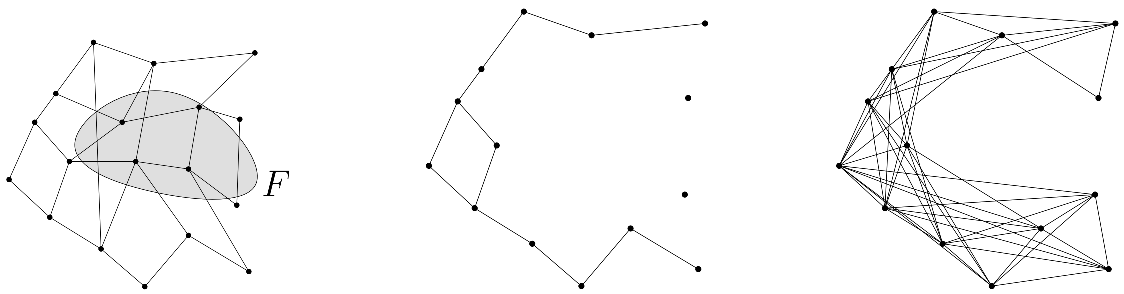

Figure 1: The input graph $\mathcal{G}$ and a fault region F, the graph $\mathcal{G}\ominus F$, and the graph $\mathcal{G}_c(P)\ominus F$.

attention to half-plane faults. This lemma will also be used in later sections. Let $\mathcal{H}$ be the family of half-planes in the plane.

PROPOSITION 2.1. *A geometric graph $\mathcal{G}$ on a set P of points in the plane is a $\mathcal{C}$-fault tolerant t-spanner if and only if it is an $\mathcal{H}$-fault tolerant t-spanner.*

Proof. Obviously a graph is $\mathcal{H}$-fault tolerant if it is $\mathcal{C}$-fault tolerant. To prove the other direction assume that $\mathcal{G}$ is an $\mathcal{H}$-fault tolerant t-spanner and that $F \in \mathcal{C}$ is an arbitrary convex area fault. We need to prove that between every pair of points $u, v \in P \setminus F$ there is a path in $\mathcal{G}\ominus F$ of length at most t times the length of the shortest path in $\mathcal{G}'_c = \mathcal{G}_c(P)\ominus F$.

If u and v are not connected in $\mathcal{G}'_c$ we are done. Otherwise, let Π be a shortest path between u and v in $\mathcal{G}'_c$. We claim that for every edge (p,q) in Π there is a path in $\mathcal{G}\ominus F$ of length at most $t \cdot d(p,q)$. Since the edge (p,q) lies outside F and F is convex, there must be a half-plane h that contains F but does not intersect (p,q). Since $\mathcal{G}$ is an $\mathcal{H}$-fault tolerant t-spanner there is a path $\Pi_{(p,q)}$ between p and q in $\mathcal{G}\ominus h$ of length at most $t \cdot d(p,q)$. Furthermore, since $F \subset h$ the path $\Pi_{(p,q)}$ also exists in $\mathcal{G}\ominus F$. The claim and, hence, the lemma follows. □

2.1 The well-separated pair decomposition.

The well-separated pair decomposition (WSPD) was developed by Callahan and Kosaraju [2]. We briefly review (the planar version of) this decomposition here. Let $s > 0$ be a real number, referred to as the *separation constant.* We say that two point sets A and B in the plane are *well-separated* with respect to s, if there are two disjoint disks D_A and D_B of the same radius, r, such that (i) D_A contains A, (ii) D_B contains B, and (iii) the distance between D_A and D_B is at least $s \cdot r$.

DEFINITION 2.1. *Let P be a set of n points in the plane and let $s > 0$ be a real number.* A well-separated pair decomposition *(WSPD) for P with respect to s is a collection $\mathcal{W} := \{(A_1, B_1), ..., (A_m, B_m)\}$ of pairs of non-empty subsets of P such that*

1. *A_i and B_i are well-separated w.r.t. s, for all $i = 1, \ldots, m$.*
2. *for any two distinct points p and q of P, there is exactly one pair (A_i, B_i) in the collection, such that (i) $p \in A_i$ and $q \in B_i$, or (ii) $q \in A_i$ and $p \in B_i$.*

The number of pairs, m, is called the *size* of the WSPD. Callahan and Kosaraju show that any set P admits a WSPD of size $m = \mathcal{O}(s^2 n)$.

2.2 Constructing a $\mathcal{C}$-fault tolerant spanner.

Callahan and Kosaraju [1] showed that the WSPD can be used to obtain a small $(1+\varepsilon)$-spanner. (Similar ideas were used earlier by Salowe [17, 18] and Vaidya [20, 21, 22].) More precisely, if one sets $s := 4 + (8/\varepsilon)$ and for any pair (A, B) in the WSPD one adds an arbitrary edge connecting a point from A to a point from B, then the result is a $(1+\varepsilon)$-spanner with $\mathcal{O}(n/\varepsilon^2)$ edges.

Unfortunately this construction is not $\mathcal{C}$-fault tolerant, because a fault F can destroy the spanner edge that connects a pair (A, B), while some other edges between A and B (which are not in the spanner) may survive the fault. Hence, we need to add more than a single edge for (A, B). Let $\mathrm{CH}(A)$ and $\mathrm{CH}(B)$ denote the convex hulls of A and B, respectively. At first sight it seems that adding the two outer tangents of $\mathrm{CH}(A)$ and $\mathrm{CH}(B)$ to our spanner may lead to a $\mathcal{C}$-fault tolerant spanner, but this is not the case either. Instead, we will triangulate the region in between the two convex hulls in an arbitrary manner, as illustrated in Fig. 2a. Let $E(A, B)$ be the set of triangulation edges added between $\mathrm{CH}(A)$ and $\mathrm{CH}(B)$, and let $\mathcal{G}$ be the obtained graph. Note that any triangulation between $\mathrm{CH}(A)$ and $\mathrm{CH}(B)$ has the same number of edges. Throughout the paper we will use the notation $|\cdot|$ to denote the number of elements in a set.

LEMMA 2.1. *The graph $\mathcal{G}$ is a $\mathcal{C}$-fault tolerant $(1+\varepsilon)$-spanner for P of size $\sum_{(A,B)\in\mathcal{W}} |E(A,B)|$.*

Proof. The size of the graph is obviously $\sum_{(A,B)\in\mathcal{W}} |E(A,B)|$, so it remains to show that it is a $\mathcal{C}$-fault tolerant $(1+\varepsilon)$-spanner. By Proposition 2.1 and the properties of the WSPD it is sufficient to show the following: Let h be a half-plane fault, let u, v be points not in h, and let (A, B) be a pair with $u \in A$ and $v \in B$; then there is an edge $e \in E(A, B)$ between $\mathrm{CH}(A)$ and $\mathrm{CH}(B)$ that is outside h.

To see this we first prove that, given a point set P and a triangulation T of P, the graph $T \ominus h$ is connected for any half-plane h. Assume without loss of generality that h is below and bounded by a horizontal line. Since any point of $P \backslash h$ not on the convex hull must have an edge connecting it to a point further away from h, we can walk from p away from h along edges of T until we reach a point on the convex hull of P. Moreover, any two convex hull points in $P \backslash h$ can be connected by convex hull edges outside h. It follows that $T \ominus h$ is indeed connected.

Now consider any triangulation T on $A \cup B$ that includes $E(A, B)$. Then $T \ominus h$ must be connected. Since $u, v \notin h$, and $u \in A$ and $v \in B$, this means there must be an edge $e \in E(A, B)$ outside h. □

2.3 Linear-size spanners for special cases. The method described above can be used to get small $\mathcal{C}$-fault tolerant spanners for several special cases. For example, if P is in convex position then $|E(A, B)| \leq 3$ for any pair (A, B) in the decomposition, so we get:

THEOREM 2.1. *For any set P of n points in convex position in the plane and any $\varepsilon > 0$, there exists a $\mathcal{C}$-fault tolerant $(1+\varepsilon)$-spanner of size $\mathcal{O}(n/\varepsilon^2)$.*

We can also get an expected small $\mathcal{C}$-fault tolerant spanner if the point set P is generated by picking n points uniformly at random in the unit square.

THEOREM 2.2. *Let P be a set of n points uniformly distributed in the unit square U. For any $\varepsilon > 0$ there is a $\mathcal{C}$-fault tolerant $(1+\varepsilon)$ spanner of expected size $\mathcal{O}(n/\varepsilon^2)$ for P.*

Proof. (Sketch) Construct a quadtree partitioning of U into smaller and smaller squares, until each square has size (side length) roughly $1/\sqrt{n}$. Thus the quadtree has $\mathcal{O}(n)$ leaves. Level ℓ of the quadtree corresponds to a regular subdivision of U into squares of size $1/2^\ell$. One can show that there exists a WSPD of size $\mathcal{O}(n/\varepsilon^2)$ for P such that each pair (A, B) either corresponds to two squares at the same level, or A and B are both singleton points that lie in nearby cells of the final subdivision.

Now consider a square σ at level ℓ. Note that the points that fall inside σ behave as if they were added uniformly at random inside σ. Hence, if n_σ denotes the number of points inside σ, then the expected size of the convex hull of those points is $\mathcal{O}(\log n_\sigma)$ [16]. Furthermore, because the area of σ is $1/2^{2\ell}$, we have $\mathrm{E}[n_\sigma] = \mathcal{O}(n/2^{2\ell})$. Using these two properties, one can show that this approach leads to a $\mathcal{C}$-fault tolerant $(1+\varepsilon)$-spanner of expected size $\mathcal{O}(n/\varepsilon^2)$. □

2.4 $\mathcal{C}$-fault tolerant Steiner spanners. Above we showed that the WSPD can be used to construct $\mathcal{C}$-fault tolerant spanners of small size when the points are in convex position or uniformly distributed. For arbitrary point sets, however, the size of the spanner may be $\Omega(n^2)$. In this section we will show that if we are allowed to add Steiner points, we can always use the above method to get a linear-size spanner:

THEOREM 2.3. *For any set P of n points in the plane and any $\varepsilon > 0$, one can construct a $\mathcal{C}$-fault tolerant Steiner $(1+\varepsilon)$-spanner of size $\mathcal{O}(n/\varepsilon^2)$ by adding at most $4(n-1)$ Steiner points.*

The idea is to add a set Q of Steiner points to P such that $|E(A, B)| = \mathcal{O}(1)$ for any pair (A, B) in the WSPD of $P \cup Q$. Then the theorem immediately follows from Lemma 2.1.

Our method is based on the WSPD construction by Fisher and Har-Peled [9]. Their construction uses a compressed quadtree, which is defined as follows.

Let U be a bounding square for P. Partition U into four equal-sized squares (quadrants). Continue recursively until each square in the final subdivision contains a single point. The process can be modeled as a quadtree $\mathcal{T}(P)$, whose leaves correspond to squares in the final subdivision and whose internal nodes correspond to squares containing more than one point (those squares were subdivided during the partitioning process). We denote the square corresponding to a node $\nu \in \mathcal{T}(P)$ by $\sigma(\nu)$, and the subset of points from P inside $\sigma(\nu)$ by $P(\nu)$.

When some of the points are very close together, a quadtree can have superlinear size. A *compressed quadtree* $\mathcal{T}^*(P)$ for P therefore removes internal nodes ν from $\mathcal{T}(P)$ for which all points from P lie in the same quadrant of $\sigma(\nu)$. A compressed quadtree has at most $n-1$ internal nodes. Fisher and Har-Peled show that one can obtain a WSPD of size $\mathcal{O}(s^2 n)$ for P that consists of pairs $(P(\nu_1), P(\nu_2))$ where ν_1 and ν_2 are nodes in $\mathcal{T}^*(P)$.

The set Q of Steiner points that we use is defined as follows. Let $\mathcal{T}^*(P)$ be a compressed quadtree for P. Without loss of generality, we may assume that no point from P lies on any of the splitting lines. For each internal node ν of $\mathcal{T}^*(P)$, we add the four corner points of $\sigma(\nu)$ to Q. To avoid degenerate cases, we slightly

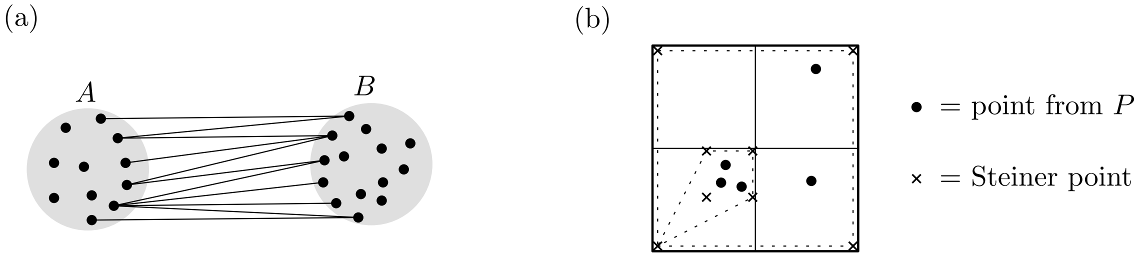

Figure 2: (a) Illustrating the construction of the WSPD-graph. (b) Illustration for the proof of Lemma 2.2.

move each point into the interior of $\sigma(\nu)$. Note that two (or more) squares $\sigma(\nu_1)$ and $\sigma(\nu_2)$ may share their top right corner, for instance. In this case we add this (slightly shifted) corner point only once. The resulting set Q has size at most $4(n-1)$. The next lemma finishes the proof of Theorem 2.3.

LEMMA 2.2. *Let $\mathcal{T}^*(\overline{P})$ be a compressed quadtree for $\overline{P} := P \cup Q$, where the initial bounding square U is the same as for $\mathcal{T}^*(\overline{P})$, and let ν be an internal node of $\mathcal{T}^*(\overline{P})$. Then $\mathrm{CH}(\overline{P}(\nu))$ has at most four vertices.*

Proof. If the square $\sigma(\nu)$ contains zero or one point from P then at most one Steiner point has been added inside $\sigma(\nu)$, and the lemma is true. If $\sigma(\nu)$ contains two or more points then there are two cases, both illustrated in Fig. 2b.

Let μ be the node of $\mathcal{T}^*(P)$ such that $P(\mu) = \overline{P}(\nu) \cap P$. Note that the four shifted corners of $\sigma(\mu)$ were added as Steiner points to Q. If $\sigma(\mu) = \sigma(\nu)$ then $\mathrm{CH}(\overline{P}(\nu))$ is a square. Otherwise, $\mathrm{CH}(\overline{P}(\nu))$ is formed by three of the four corners of $\sigma(\mu)$ together with the unique corner of $\sigma(\nu)$ that generated a Steiner point at some ancestor of ν in $\mathcal{T}^*(P)$, see Fig. 2b. Hence, in this case $\mathrm{CH}(\overline{P}(\nu))$ has four vertices as well. □

3 $\mathcal{C}$-fault tolerant spanners for arbitrary point sets

In this section we consider the problem of constructing a sparse $\mathcal{C}$-fault tolerant $(1+\varepsilon)$-spanner for an arbitrary set P of n points in the plane without using Steiner points. The method based on the WSPD that was described in the previous section does not guarantee a small spanner in general. Here we will describe a method that is guaranteed to result in a spanner of size $\mathcal{O}(n \log n)$.

Throughout this section $d(\cdot,\cdot)$ denotes the shortest distance between two objects (points, disks, etc.), $\mathrm{radius}(D)$ denotes the radius of a disk D, and D_A denotes the smallest enclosing disk of a set A.

3.1 SSPDs and fault-tolerant spanners. The problem with the WSPD in our application is that, even though the number of pairs in the WSPD is $\mathcal{O}(n)$, the total number of points over all the pairs can be $\Theta(n^2)$. Therefore we will introduce a relaxed version of the WSPD, the SSPD.

Let A and B be two sets of points in the plane, and let $s > 0$ be a constant. We say that A and B are *semi-separated with respect to separation constant s* if $d(D_A, D_B) \geq s \cdot \min(\mathrm{radius}(D_A), \mathrm{radius}(D_B))$.

Thus we allow the disks D_A and D_B to be of different sizes and we only require that the distance between the disks is large relative to the smaller disk. Note that using the same notations we can reformulate the definition of well-separated with respect to s as $d(D_A, D_B) \geq s \cdot \max(\mathrm{radius}(D_A), \mathrm{radius}(D_B))$.

We now define our SSPD.

DEFINITION 3.1. *Let P be a set of n points in the plane and let $s > 0$ be a real number. A* semi-separated pair decomposition *(SSPD) for P w.r.t. s is a collection $\{(A_1, B_1), \ldots, (A_m, B_m)\}$ of pairs of non-empty subsets of P such that*

1. *A_i and B_i are semi-separated w.r.t. s, for all $i = 1, \ldots, m$.*
2. *for any two distinct points p and q of P, there is exactly one pair (A_i, B_i) in the collection, such that (i) $p \in A_i$ and $q \in B_i$, or (ii) $q \in A_i$ and $p \in B_i$,*

The *weight* of a set A is defined as the number of points in A, the weight of a semi-separated pair (A, B) is the sum of the weights of A and B, and the weight of an SSPD is the total weight of all the pairs. Later we will prove that it is possible to compute an SSPD of weight $\mathcal{O}(n \log n)$. First, however, we will show how to use the SSPD to obtain a $\mathcal{C}$-fault tolerant spanner. The idea is to add edges to the spanner for each pair in the SSPD. Because the pairs in an SSPD are only semi-separated, however, adding a single edge for every pair does not necessarily lead to a good spanner. Therefore

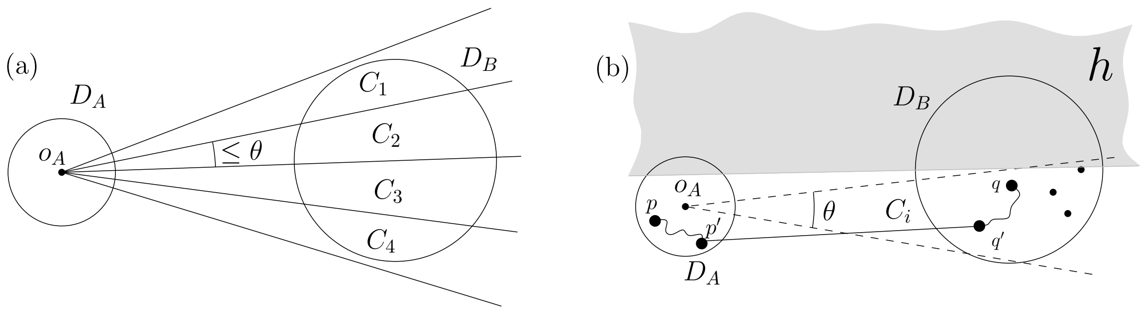

Figure 3: (a) The cones of angle at most θ defined with respect to o_A and $P(v)$. (b) Illustrating the proof of Lemma 3.3.

we use an idea that is also used in the construction of Θ-graphs [4, 11].

Consider a pair (A, B) in an SSPD for P. Then $d(D_A, D_B) \geq s \cdot \min(\text{radius}(D_A), \text{radius}(D_B))$. Assume without loss of generality that $\text{radius}(D_A) \leq \text{radius}(D_B)$, and let o_A denote the center of D_A— see Fig. 3a. The set $E(A, B)$ of edges added to the spanner for the pair (A, B) is found as follows.

1. Partition the plane into $k := \lceil 2\pi/\theta \rceil$ cones $C_1, \ldots, C_k$, all with apex at o_A and with interior angle at most θ, where θ is a suitable constant to be specified later. Let $B^{(i)} := B \cap C_i$ denote the subset of points from B inside the cone C_i; here we assume without loss of generality that no point lies on the boundary between two cones.
2. Let $\text{CH}(A)$ be the convex hull of A. For each $B^{(i)}$, we sort the points in $B^{(i)}$ in order of increasing distance to o_A. Let $q_1, q_2, \ldots$ denote the sorted list of points. We process each point q_j in order as follows. Let $\text{CH}(A')$ be the current convex hull, which is the convex hull of the set $A' = A \cup \{q_1, \ldots, q_{j-1}\}$. We add all edges between q_j and vertices of A on $\text{CH}(A')$ for which the edge does not intersect $\text{CH}(A')$. Next we update $\text{CH}(A')$ by adding the point q_j.

By construction the edges that we are adding to $E(A, B)$ do not cross, so we have:

Lemma 3.1. $|E(A, B)| = \mathcal{O}(|A| + |B|)$.

To prove that our approach will result in a fault-tolerant spanner, we need the following lemma. Consider the ordered set $B^{(i)}$ of the points in B inside the cone C_i.

Lemma 3.2. *Let h be a half-plane fault such that both A and $B^{(i)}$ have at least one point outside h. Of all the points in $B^{(i)}$ outside h, let q_j be the one with minimum distance to o_A. Then there is an edge in $E(A, B)$ connecting q_j to a point $p_k \in A$ outside h.*

Proof. By assumption, q_j is outside h and there is at least one point from A outside h. On the other hand, by the choice of q_j, the points $q_1, \ldots, q_{j-1}$ are all inside h. This implies that the edges added from those points cannot completely hide all the points in $A \setminus h$ from q_j. Hence, when q_j is processed, at least one edge to a point in $A \setminus h$ is added. □

Let $\mathcal{G}$ be the graph obtained after applying the above procedure to every pair (A, B) in the SSPD for the point set P. Next we prove that $\mathcal{G}$ is a $(1 + \varepsilon)$-spanner if we choose the separation constant s and the angle θ suitably and, moreover, that it is $\mathcal{C}$-fault tolerant. For this we will need the following condition on the structure of the SSPD (which will be satisfied by the SSPD we will construct later).

Monotonicity condition: Suppose p, q are two points that are in the same set X of some pair (A_i, B_i) of the SSPD—thus $X = A_i$ or $X = B_i$—and let (A_j, B_j) be the unique pair in the SSPD such that $p \in A_j$ and $q \in B_j$, or $p \in B_j$ and $q \in A_j$. Then the weights of A_j and B_j are both less than the weight of X.

Lemma 3.3. *If the separation constant s of the SSPD is taken as $s := \frac{3t+1}{\alpha t - 1}$ where $\alpha = \cos\theta - \sin\theta > 1/t$, then the graph $\mathcal{G}$ is an $\mathcal{H}$-fault tolerant t-spanner.*

Proof. Let h be an arbitrary half-plane. To prove the lemma we must show that for each pair of points $p, q \in P$ outside h there is a t-path connecting them in $\mathcal{G} \ominus h$. According to the definition of the SSPD there exists a semi-separated pair (A, B) such that $p \in A$ and $q \in B$ (or vice versa).

The proof is done by induction on the maximum weight of A and B.

Base case: If the maximum weight of A and B is 1, then both sets are singletons and therefore we must have an edge between them.

Induction step: Suppose the maximum weight of A and B is k, for some $k > 1$, and assume the lemma holds for all points in pairs whose maximum weight is less than k.

Assume without loss of generality that $\text{radius}(D_A) \leq \text{radius}(D_B)$, and let $o = o_A$ denote the center of D_A. Let C_i be the cone with apex o that contains q. Let q' be the point in $B^{(i)} \setminus h$ closest to o. According to Lemma 3.2 there is an edge between q' and some point p' in A outside h—see Fig. 3b. By the induction hypothesis, which we may apply because of the monotonicity condition, there are t-paths from p to p' and from q' to q in $\mathcal{G} \ominus h$. By connecting these paths using the edge (p', q') we obtain a path Π in $\mathcal{G} \ominus h$. Next we prove that Π is a t-path between p and q. Set $r := \text{radius}(D_A)$. Consider the triangle $\triangle oqq'$. Since $\angle qoq' \leq \theta$, we have $d(q, q') \leq d(o, q) - (\cos\theta - \sin\theta) \cdot d(o, q')$. The total length of Π, denoted $\text{length}(\Pi)$, can now be bounded as follows:

$$
\begin{aligned}
\text{length}(\Pi) &\leq t \cdot d(p, p') + d(p', q') + t \cdot d(q, q') \\
&\leq 2rt + (r + d(o, q')) \\
&\quad + t \cdot (d(o, q) - (\cos\theta - \sin\theta) \cdot d(o, q')) \\
&\leq 2rt + (r + d(o, q')) + t(d(o, q) - r) \\
&\quad + tr - t\alpha \cdot d(o, q') \\
&\leq 3rt + (r + d(o, q')) + t \cdot d(p, q) \\
&\quad - t\alpha \cdot d(o, q') \\
&\leq t \cdot d(p, q) + r(3t + 1) \\
&\quad + (1 - t\alpha) \cdot d(o, q').
\end{aligned}
$$

Since $d(D_A, D_B) \geq s \cdot r$, we have $d(o, q') \geq s \cdot r$. Hence, since $\alpha > 1/t$ we get

$$
\begin{aligned}
\text{length}(\Pi) &\leq t \cdot d(p, q) + r(3t + 1) + sr(1 - t\alpha) \\
&\leq t \cdot |pq|.
\end{aligned}
$$

This completes the proof of the lemma. □

By combining Proposition 2.1 and Lemmas 3.1 and 3.3 with the construction algorithm presented below for constructing an SSPD of weight $\mathcal{O}(n \log n)$, we get the following theorem.

THEOREM 3.1. *For any set P of n points in the plane and any $\varepsilon > 0$, there exists a $\mathcal{C}$-fault tolerant $(1+\varepsilon)$-spanner of P with $\mathcal{O}(n \log n)$ edges.*

3.2 Computing an SSPD.

To compute an SSPD for a given point set P, we use a BAR-tree, as introduced by Duncan *et al.* [7]. A BAR-tree for a point set P is a BSP-tree with the following properties: (i) each leaf region contains at most one point from P, (ii) the tree has size $\mathcal{O}(n)$, (iii) if we go down two levels in the tree then the size of the subtree reduces with a factor of β, for some constant $1/2 < \beta < 1$, so its depth is $\mathcal{O}(\log n)$, (iv) the region $R(\nu)$ associated with an (internal or leaf) node ν has bounded aspect ratio, that is, the smallest enclosing circle of $R(\nu)$ is only a constant times larger than the largest inscribed circle. Moreover, BAR-trees only use splitting lines that are horizonal, vertical, or diagonal, but this will not be relevant for our discussion.

Let $\mathcal{T}$ be a BAR-tree on the point set P. For a node ν, we use $\text{pa}(\nu)$ to denote the parent of ν, and we use $P(\nu)$ to denote the subset of points from P that are stored in the leaves of the subtree $\mathcal{T}_\nu$ rooted at ν. The *weight* of a node ν is the number of points in $P(\nu)$, and is denoted $|P(\nu)|$. We say that a node ν in $\mathcal{T}$ has *weight class* ℓ, for some integer ℓ, if and only if $|P(\nu)| \leq n/2^\ell$ and $|P(\text{pa}(\nu))| > n/2^\ell$. We denote the collection of nodes of weight class ℓ by $N(\ell)$. Obviously we have $\mathcal{O}(\log n)$ weight classes. Note that some of the nodes in the tree may not be in any weight class; this can happen when the weight of a node ν is almost the same as the weight of its parent. (For example, this happens when $|P(\text{pa}(\nu))| = n/2^\ell$ for some ℓ and $|P(\nu)| = n/2^\ell - 1$.) It can also happen that a node belongs to more than one weight class, namely when the weight of a node is much smaller than the weight of its parent. The following lemma is straightforward.

LEMMA 3.4. *Every leaf node is in weight class $\ell_{\max}$, where $\ell_{\max} = \lfloor \log n \rfloor$. Furthermore, on any root-to-leaf-path there is exactly one node with weight class ℓ, for any $0 \leq \ell \leq \ell_{\max}$.*

For a node $\nu \in N(\ell)$, we define its *ℓ-parent* to be the node $\nu' \in N(\ell - 1)$ that is on the path from the root of $\mathcal{T}$ to ν (including ν itself). We denote the ℓ-parent of ν by $\text{pa}(\ell, \nu)$. Observe that ν can be its own ℓ-parent, namely when $\nu \in N(\ell)$ and $\nu \in N(\ell - 1)$. By Lemma 3.4, if $\nu \in N(\ell)$ then one of its ancestors (possibly itself) must be in weight class $\ell - 1$, so it must have an ℓ-parent. We define $\mathcal{S}$ to be the collection of all pairs $(P(\nu), P(\mu))$ that satisfy the following three conditions:

1. ν and μ are in the same weight class, $N(\ell)$;
2. $P(\nu)$ and $P(\mu)$ are semi-separated (w.r.t. the given separation constant s);
3. $P(\text{pa}(\ell, \nu))$ and $P(\text{pa}(\ell, \mu))$ are not semi-separated.

LEMMA 3.5. *$\mathcal{S}$ is an SSPD for P with respect to s.*

Proof. By construction any pair in $\mathcal{S}$ is semi-separated, so it remains to verify that for every pair of points p, q there is a unique pair $(P(\nu), P(\mu))$ such that $p \in P(\nu)$ and $q \in P(\mu)$, or vice-versa, that satisfies conditions 1–3 above.

For any $0 \le \ell \le \ell_{\max}$, define $\nu(p, \ell)$ and $\nu(q, \ell)$ to be the nodes of $N(\ell)$ on the search path to p and q, respectively. Observe that these nodes exist and are uniquely defined by Lemma 3.4. We have $\nu(p, 0) = \nu(q, 0) = \mathrm{root}(\mathcal{T})$, so the sets $P(\nu(p, 0))$ and $P(\nu(q, 0))$ are the same and therefore not semi-separated. On the other hand, $\nu(p, \ell_{\max})$ and $\nu(q, \ell_{\max})$ are leaves, and so the sets $P(\nu(p, \ell_{\max}))$ and $P(\nu(q, \ell_{\max}))$ are singletons and therefore semi-separated. It follows that there must indeed be a pair of nodes ν, μ satisfying conditions 1–3 above. Because $P(\mathrm{pa}(\ell, \nu))$ and $P(\mathrm{pa}(\ell, \mu))$ are not semi-separated by definition, it is easy to show that the pair ν, μ is unique. □

To bound the weight of the SSPD, we first prove two auxiliary lemmas.

LEMMA 3.6. *A node ν in $\mathcal{T}$ can be ℓ-parent of at most a constant number of nodes in $\mathcal{T}$.*

Proof. Consider a node $\nu \in N(\ell - 1)$ and let ν' be a node such that $\nu = \mathrm{pa}(\ell, \nu')$. Then ν' is a node in $\mathcal{T}_\nu$ (the subtree of $\mathcal{T}$ rooted at ν) in weight class ℓ. Note that no other node than ν' in $\mathcal{T}_{\nu'}$ can have ν as its ℓ-parent. Recall that the weight of a node reduces with a factor of β when we go down two levels in a BAR-tree. Since $\nu' \in N(\ell)$, its (normal) parent has weight more than $n/2^\ell$. On the other hand $\nu \in N(\ell - 1)$, so the weight of ν is at most $n/2^{\ell-1}$. Hence, the path between ν and ν' consists of at most $2k$ links, where $\beta^k = 1/2$. It follows that the total number of nodes in $\mathcal{T}$ that have ν as a ℓ-parent is bounded by 2^{2k}, which is a constant since k is a constant. □

LEMMA 3.7. *Let $0 \le \ell \le \ell_{\max}$, let $s > 0$ be a constant and let $\overline{\mathcal{S}}(\ell)$ be the set of all pairs (ν, η) such that $\nu, \eta \in N(\ell)$ and (ν, η) is not semi-separated w.r.t. s. Then*

$$\sum_{(\nu,\eta)\in\overline{\mathcal{S}}(\ell)} (|P(\nu)| + |P(\eta)|) = \mathcal{O}(n).$$

Proof. We reorder the nodes in the pairs (ν, η) such that $\mathrm{radius}(D_{P(\nu)}) \le \mathrm{radius}(D_{P(\eta)})$. (Recall that D_A denotes the smallest enclosing disk of a set A.) We claim that ν appears in a constant number of pairs as the first element of the pair. To show this let (ν, η) be an arbitrary pair. Consider a disk D of radius $s \cdot \mathrm{radius}(D_{P(\nu)})$ and with the same center as $D_{P(\nu)}$. Since (ν, η) are not semi-separated w.r.t. s, the disk $D_{P(\eta)}$ intersects D. Because of the fatness of the regions, this can happen for only a constant number of nodes η. Since $|N(\ell)| = \mathcal{O}(2^\ell)$, we can have $\mathcal{O}(2^\ell)$ nodes as the first element of the pair so the total number of pairs (ν, η) in the summation is $\mathcal{O}(2^\ell)$. The lemma follows since $|P(\nu)| \le n/2^\ell$ for each $\nu \in N(\ell)$. □

Now we are finally ready to bound the size of $\mathcal{S}$.

LEMMA 3.8. *For the SSPD $\mathcal{S}$ defined above we have*

$$\sum_{(\nu,\eta)\in\mathcal{S}} (|P(\nu)| + |P(\eta)|) = \mathcal{O}(n \log n).$$

Proof. Since the number of levels in $\mathcal{T}$ is $\mathcal{O}(\log n)$ it suffices to prove that for every fixed ℓ it holds that:

$$\sum_{\substack{(\nu,\eta)\in\mathcal{S}\\ \nu,\eta\in N(\ell)}} (|P(\nu)| + |P(\eta)|) = \mathcal{O}(n). \tag{3.1}$$

Obviously $|P(\nu)| \le |P(\mathrm{pa}(\ell, \nu))|$ for each node ν, so we can bound (3.1) by

$$\sum_{\substack{(\nu,\eta)\in\mathcal{S}\\ \nu,\eta\in N(\ell)}} (|P(\mathrm{pa}(\ell, \nu))| + |P(\mathrm{pa}(\ell, \eta))|). \tag{3.2}$$

From the definition of $\mathcal{S}$ we know that $P(\mathrm{pa}(\ell, \nu))$ and $P(\mathrm{pa}(\ell, \eta))$ are not semi-separated w.r.t. s. Furthermore, by Lemma 3.6 each node can be an ℓ-parent of a constant number of nodes. Hence, (3.2) can be bounded by

$$\sum_{(\nu,\eta)\in\overline{\mathcal{S}}(\ell-1)} \mathcal{O}(|P(\nu)| + |P(\eta|), \tag{3.3}$$

where $\overline{\mathcal{S}}(\ell - 1)$ is the set of all pair (ν, η) such that $\nu, \eta \in N(\ell - 1)$ and (ν, η) is not semi-separated w.r.t. s. According to Lemma 3.7 summation (3.3) is $\mathcal{O}(n)$, which completes the proof of the lemma. □

The following theorem summarizes the results on the SSPD construction.

THEOREM 3.2. *Given a set P of n points in the plane and a constant $s > 0$ we can compute an SSPD w.r.t. s of total size $\mathcal{O}(n \log n)$.*

4 Fault-tolerant geodesic spanners

We now consider the problem of constructing geodesic fault tolerant spanners for a set P of n points in the plane. Here we require that between any two points $u, v \in P$ outside the region fault F, there is a path in $\mathcal{G} \ominus F$ whose length is at most t times the geodesic distance between u and v in $\mathbb{R}^2 \setminus F$. As remarked in the introduction, finite size geodesic fault tolerant spanners do not exist unless we are allowed to use Steiner points, and even then they do not exist for arbitrary convex fault regions. Hence, we restrict the faults to the family $\mathcal{D}$ of disks in the plane.

Our method works as follows. We first augment P with a set of $4(n - 1)$ Steiner points as described in

Section 2. This way we can get an $\mathcal{O}(n/\varepsilon^2)$ size WSPD consisting of pairs (A, B) where the convex hull of each A and B has at most four vertices. Now fix a pair (A, B). For every pair of points u, v, where u is a vertex of $\mathrm{CH}(A)$ and v is a vertex of $\mathrm{CH}(B)$, we will add a collection of $\mathcal{O}(1/\varepsilon^3)$ Steiner points with $\mathcal{O}(1/\varepsilon^4)$ edges between them, to ensure the following: whenever both u and v are outside the fault disks D, there is a path connecting u and v through those Steiner points and outside D whose length is at most $(1+\varepsilon)$ times the geodesic distance between u and v. This is sufficient because whenever there are points $p \in A$ and $q \in B$ outside D, there are convex-hull points $u \in \mathrm{CH}(A)$ and $v \in \mathrm{CH}(B)$ outside D. Hence, we can go from p to u with a short path (by induction), then from u to v (by construction), and then from v to q (by induction).

Next we describe how to add Steiner points for the pair u, v. Without loss of generality we assume u and v are on a horizontal line at distance 1. Consider a unit square placed such that uv partitions it into two equal halves. We partition this square into a regular $(1/\varepsilon) \times (1/\varepsilon)$ grid whose cells have size $\varepsilon \times \varepsilon$—we assume for simplicity that $1/\varepsilon$ is an integer—and we put $1/\varepsilon^2$ equally-spaced Steiner points on each grid line, as shown in Fig. 4a. Notice that each grid cell has $1/\varepsilon$ Steiner points on each of its sides, and that we have $\mathcal{O}(1/\varepsilon^3)$ Steiner points in total. For each grid cell we add edges between every pair of Steiner points on its boundary, thus adding $O(1/\varepsilon^2)$ edges per cell and $O(1/\varepsilon^4)$ edges in total.

It remains to prove that we can always get a path whose length is close to the geodesic distance when we have a disk fault D. This can be argued as follows. Assume without loss of generality that the center of D lies below (or on) the line through u and v. Then the geodesic between u and v will go around D on the top side. Let D' be the disk with the same center as D but with radius $r + \varepsilon\sqrt{2}$, where r is the radius of D'. Then a grid cell that intersects $\partial D'$ (the boundary of D') does not intersect the interior of D. The geodesic from u to v consists of a straight line segment connecting u to some point x on ∂D, followed by a circular arc along ∂D from x to some point y, followed by a straight line segment connecting y to v. Draw two rays from O, the common center of D and D', through x and y, and let x' and y' denote the points where these rays intersect $\partial D'$. Next, draw the lines tangent to D' at x' and y'—these are parallel to ux and yv, respectively—and let u' and v' be the intersection of these lines with the vertical lines through u and v—see Fig. 4b. Finally, define Π to be the path consisting of the segments uu' and $u'x'$, followed by the arc along $\partial D'$ from x' to y', followed by the segments $y'v'$ and $v'v$. We have $\mathrm{length}(\Pi) \leq \left(1 + (\pi\sqrt{2} + 2\sqrt{2})\varepsilon\right)\gamma$, where γ is the geodesic distance between u and v.

From Π we construct a path Π' that uses the edges in our spanner. To this end, let $p_1, p_2, \ldots, p_k$ denote the intersection points of Π with the grid lines, ordered from u to v. Note that $u = p_1$ and $v = p_k$. For each intersection point p_i, let p'_i be the closest Steiner point above or on Π on the same grid line. We define Π' to be the path through $p'_1, \ldots, p'_k$. Since each edge in Π' is inside a grid cell intersected by $\partial D'$, the path Π' does not intersect D. Moreover, the length of Π' inside a grid cell cannot exceed the length of Π inside that cell by more than $2\varepsilon^2$. Hence, $\mathrm{length}(\Pi') \leq \mathrm{length}(\Pi) + 4\varepsilon$. Putting everything together, we can get the following theorem.

Theorem 4.1. *For any set P of n points and any $\varepsilon > 0$, there exists a $\mathcal{D}$-fault tolerant geodesic Steiner $(1+\varepsilon)$-spanner of P with $\mathcal{O}(n/\varepsilon^6)$ edges in $\mathcal{O}(n/\varepsilon^5)$ points.*

5 Concluding Remarks

We introduced the concept of region-fault tolerant spanners for planar point sets, and proved the existence of region-fault tolerant spanners of small size. In the full paper we show that all constructions can be carried out in $\mathcal{O}(n \log^2 n)$ time or less. The main open problem is to determine whether our $\mathcal{O}(n \log n)$ bound on the spanner size for arbitrary point sets can be improved to $\mathcal{O}(n)$.

Our spanner construction for arbitrary point sets uses the SSPD, a relaxation of the WSPD. A similar variant with weaker properties, the SSD, was introduced by Varadarajan [23] who used it to solve min-cost perfect matching for points in the plane. His algorithm runs in $\sqrt{n}$ phases where each phase takes time proportional to the weight of the SSD plus the time it takes to compute the SSD. Since our SSPD satisfies the conditions of Varadarajan's SSD, and we can compute it more quickly, we can improve the running time the min-cost perfect matching algorithm to $\mathcal{O}(n^{3/2} \log^2 n)$.

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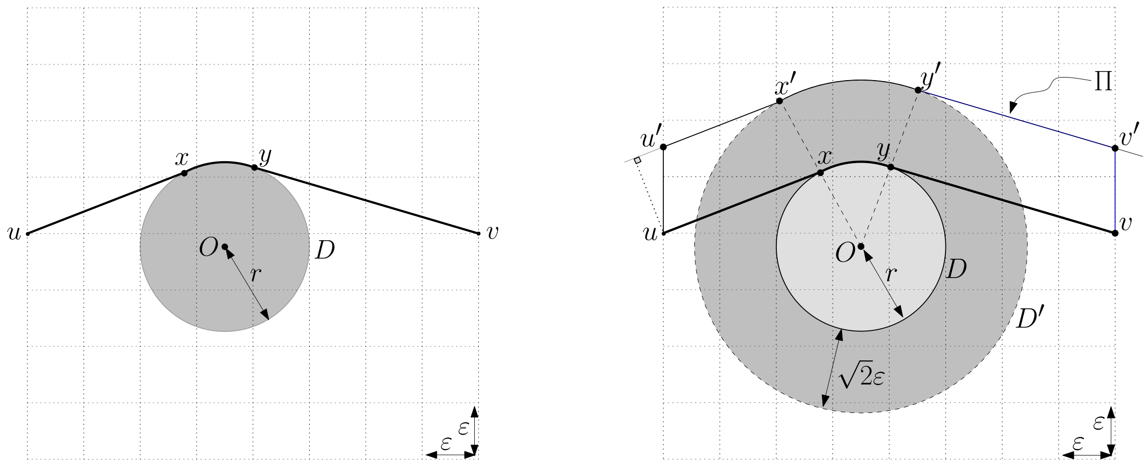

Figure 4: (a) The Steiner points added for u, v. (b) The path Π.

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