New and Improved Spanning Ratios for Yao Graphs

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ABSTRACT

For a set of points in the plane and a fixed integer $k > 0$, the Yao graph Y_k partitions the space around each point into k equiangular cones of angle $\theta = 2\pi/k$, and connects each point to a nearest neighbor in each cone. It is known for all Yao graphs, with the sole exception of Y_5, whether or not they are geometric spanners. In this paper we close this gap by showing that for odd $k \geq 5$, the spanning ratio of Y_k is at most $1/(1 - 2\sin(3\theta/8))$, which gives the first constant upper bound for Y_5, and is an improvement over the previous bound of $1/(1 - 2\sin(\theta/2))$ for odd $k \geq 7$. We further reduce the upper bound on the spanning ratio for Y_5 from 10.9 to $2 + \sqrt{3} \approx 3.74$, which falls slightly below the lower bound of 3.79 established for the spanning ratio of Θ_5 (Θ-graphs differ from Yao graphs only in the way they select the closest neighbor in each cone). This is the first such separation between a Yao and Θ-graph with the same

number of cones. We also give a lower bound of 2.87 on the spanning ratio of Y_5. Finally, we revisit the Y_6 graph, which plays a particularly important role as the transition between the graphs ($k > 6$) for which simple inductive proofs are known, and the graphs ($k \leq 6$) whose best spanning ratios have been established by complex arguments. Here we reduce the known spanning ratio of Y_6 from 17.6 to 5.8, getting closer to the spanning ratio of 2 established for Θ_6.

1. INTRODUCTION

The complete Euclidean graph defined on a point set S in the plane is the graph with vertex set S and edges connecting each pair of points in S, where each edge xy has as weight the Euclidean distance $|xy|$ between its endpoints x and y. Although this graph is useful in many different contexts, its main disadvantage is that it has a quadratic number of edges. As such, much effort has gone into the development of various methods for constructing graphs that *approximate* the complete Euclidean graph. What does it mean to approximate this graph? One standard approach is to construct a spanning subgraph with fewer edges (typically linear) with the additional property that every edge e of the complete Euclidean graph is approximated by a path in the subgraph whose weight is not much more than the weight of e. This gives rise to the notion of a *t-spanner*. A *t*-spanner of the complete Euclidean graph is a spanning subgraph with the property that for each pair of vertices x and y, the weight of a shortest path in the subgraph between x and y is at most $t \geq 1$ times $|xy|$. The *spanning ratio* is the smallest t for which the subgraph is a *t*-spanner. Span-

ners find many applications, such as approximating shortest paths or minimum spanning trees. For a comprehensive overview of geometric spanners and their applications, we refer the reader to the book by Narasimhan and Smid [12].

One of the simplest ways of constructing a t-spanner is to first partition the plane around each vertex x into a fixed number of cones[1] and then add edges connecting x to a closest vertex in each cone. Intuition suggests that this would yield a graph whose spanning ratio depends on the number of cones. Indeed, this is one of the first approximations of the complete Euclidean graph, referred to as *Yao graphs* in the literature, introduced independently by Flinchbaugh and Jones [11] and Yao [13]. We denote the Yao graph by Y_k where k is the number of cones, each having angle $\theta = 2\pi/k$. Yao used these graphs to simplify computation of the Euclidean minimum spanning tree. Flinchbaugh and Jones studied their graph theoretic properties. Neither of them actually proved that they are t-spanners.

To the best of our knowledge, the first proof that Yao graphs are spanners was given by Althöfer *et al.* [1]. They showed that for every $t > 1$, there exists a k such that Y_k is a t-spanner. It appears that some form of this result was known earlier, as Clarkson [8] already remarked in 1987 that Y_{12} is a $1 + \sqrt{3}$-spanner, albeit without providing a proof or a reference. Bose *et al.* [6] provided a more specific bound on the spanning ratio, by showing that for $k > 8$, Y_k is a geometric spanner with spanning ratio at most $1/(\cos\theta - \sin\theta)$. This was later strengthened to show that for $k > 6$, Y_k is a $1/(1-2\sin(\theta/2))$-spanner [4]. Damian and Raudonis [9] showed that Y_6 is a 17.64-spanner, and Bose *et al.* [5] showed that Y_4 is a 663-spanner. For $k < 4$, El Molla [10] showed that there is no constant t such that Y_k is a t-spanner. This leaves open only the question of whether Y_5 is a constant spanner.

In this paper we close this gap by showing that for odd $k \geq 5$, the spanning ratio of Y_k is at most $1/(1-2\sin(3\theta/8))$. This gives the first constant upper bound for Y_5 and implies that Y_k is a constant spanner for all $k \geq 4$. For odd $k \geq 7$, our result also improves on the previous bound of $1/(1-2\sin(\theta/2))$. A more careful analysis allows us to reduce the upper bound on the spanning ratio of Y_5 from 10.9 to $2 + \sqrt{3} \approx 3.74$. We also give a lower bound of 2.87 on the spanning ratio of Y_5. This complements a recent result on the spanning ratio of Θ_5, which differs from Y_5 only in the distance measure it uses to select the closest neighbor in each cone: instead of Euclidean distance, it projects each vertex on the bisector of the cone and selects the vertex with the closest projection. Bose *et al.* [7] showed that Θ_5 has a spanning ratio in the interval $[3.79, 9.96]$. Because our upper bound of 3.74 on the spanning ratio of Y_5 is slightly lower than the lower bound of 3.79 on the spanning ratio of Θ_5, this result establishes the first separation between the spanning ratio of Yao and Θ-graphs. For all other $k \geq 4$, it is unclear which of Θ_k or Y_k has a better spanning ratio.

Finally, we revisit the Y_6 graph, which plays a particularly important role as the transition between the graphs ($k > 6$) for which simple inductive proofs are known, and the graphs ($k \leq 6$) whose best spanning ratios are established by complex arguments. Here we reduce the known spanning ratio of Y_6 from 17.64 to 5.8, thus moving toward the spanning ratio of 2 established for Θ_6 [3]. In contrast to Y_5, we also present

a lower bound of 2 on the spanning ratio of Y_6, showing that it can never improve upon Θ_6 in this regard.

2. SPANNING RATIO OF Y_k, FOR ODD k

In this section we study the spanning properties of the Yao graphs Y_k defined on a plane point set S by an odd number of cones $k \geq 5$, each of angle $\theta = 2\pi/k$. For $k = 5$ in particular, this is the first result showing that Y_5 is a constant spanner. For odd values $k > 5$, we improve the currently known bound on the spanning ratio of Y_k.

We start with a few definitions. For a fixed k, let $Q_i(a)$ be the half-open cone of angle $2\pi/k$ with apex a, including the angle range $[i, i+1) \cdot 2\pi/k$, for $i = 0, \ldots, k-1$, where angles are measured counterclockwise from the positive x-axis. The directed graph $\overrightarrow{Y_k}$ includes exactly one directed edge from a to a closest point in $Q_i(a)$, for each $i = 0, \ldots, k-1$. If there are several equally-closest points within $Q_i(a)$, then ties are broken arbitrarily. The graph Y_k is the undirected version of $\overrightarrow{Y_k}$. For any two points $a, b \in S$, let $p(a,b)$ denote the length of a shortest path in Y_k from a to b.
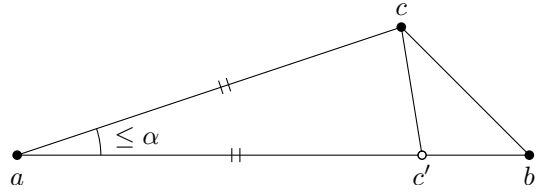

Figure 1: If α is small, there is a close relation between $|ac|$ and $|bc|$.

LEMMA 1. *Given three points a, b, and c, such that $|ac| \leq |ab|$ and $\angle bac \leq \alpha < \pi$, then*

$$|bc| \leq |ab| - (1 - 2\sin(\alpha/2)) |ac|.$$

PROOF. Let c' be the point on ab such that $|ac| = |ac'|$ (see Fig. 1). Since acc' forms an isosceles triangle,

$$|cc'| = 2\sin(\angle bac/2)|ac| \leq 2\sin(\alpha/2)|ac|.$$

Now, by the triangle inequality,

$$\begin{aligned} |bc| &\leq |cc'| + |c'b| \\ &\leq 2\sin(\alpha/2)|ac| + |ab| - |ac'| \\ &= |ab| - (1 - 2\sin(\alpha/2))|ac|. \end{aligned}$$

$\square$

THEOREM 2. *For any odd integer $k \geq 5$, Y_k has spanning ratio at most $t = 1/(1 - 2\sin(3\theta/8))$.*

PROOF. Let $a, b \in S$ be an arbitrary pair of points. We show that there is a path in Y_k from a to b no longer than $t|ab|$. For simplicity, let $Q(a)$ denote the cone with apex a that contains b, and let $Q(b)$ denote the cone with apex b that contains a. Rotate the point set S such that the bisector of $Q(a)$ is in the direction of the positive y-axis, as depicted in Fig. 2. Assume without loss of generality that b lies to the right of this bisector; the case when b lies to the left of this bisector is symmetric.

Let α be the angle formed by the segment ab with the bisector of $Q(a)$, and let β be the angle formed by ab with the bisector of $Q(b)$. Since k is odd, the bisector of $Q(a)$ is
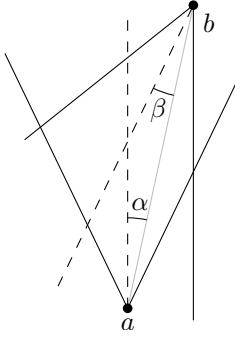
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[1]The orientation of the cones is the same for all vertices.

Figure 2: Since opposite cones are not symmetric, either α or β is small.

parallel to the right boundary of $Q(b)$. Hence, we have that $\alpha = \theta/2 - \beta$. Assume without loss of generality that α is the smaller of these two angles (if not, we exchange the roles of a and b). It follows that $\alpha \le \theta/4$.

Our proof is by induction on the distance $|ab|$. In the base case $|ab|$ is minimal among all distances between pairs of points, which means that there is no point $c \in Q(a)$ that is strictly closer to a than b. Therefore either $ab \in Y_k$, in which case our proof for the base case is finished, or there is a point $c \in Q(a)$ such that $|ab| = |ac|$ and $ac \in Y_k$. In this latter case, since $\alpha \le \theta/4$ and $k \ge 5$, the angle between ab and ac is at most $\theta/2 + \alpha \le 3\theta/4 \le 3/4 \cdot (2\pi/5) = 3\pi/10$. This is less than $\pi/3$, which implies that $|bc| < |ab|$. This contradicts our assumption that $|ab|$ is minimal. It follows that $\overrightarrow{ab} \in \overrightarrow{Y_k}$ and the base case holds.

For the inductive step, let $c \in Q(a)$ be such that $\overrightarrow{ac} \in \overrightarrow{Y_k}$. If c coincides with b, then $p(a, b) = |ab|$ and the proof is finished. So assume that $c \ne b$. Because c is the closest vertex to a in this cone, and because $\angle cab \le \theta/2 + \alpha \le 3\theta/4$, we can apply Lem. 1 to derive $|cb| \le |ab| - (1 - 2\sin(3\theta/8))|ac| = |ab| - |ac|/t$, which is strictly less than $|ab|$. Thus we can use the inductive hypothesis on cb to determine a path between a and b of length

$$p(a,b) \;\le\; |ac| + t|cb| \;\le\; |ac| + t\left(|ab| - \frac{|ac|}{t}\right) \;=\; t|ab|.$$

$\square$

Applying this result to Y_5 yields a spanning ratio of $1/(1 - 2\sin(3\pi/20)) \approx 10.868$. This is the first known upper bound on the spanning ratio of Y_5 and fully settles the question of which Yao graphs are spanners.

COROLLARY 3. *The graph Y_k is a spanner if and only if $k \ge 4$.*

Next we lower the upper bound on the spanning ratio of Y_5 by taking a closer look at all feasible configurations.

THEOREM 4. *The graph Y_5 has spanning ratio at most $2 + \sqrt{3} \approx 3.74$.*

Here we also use induction on the pairwise distances between pair of points in S. Consider the same configuration used in the proof of Thm. 2: $a \in Q(b)$ and $b \in Q(a)$ are points in S, and we seek a short path from a and b; the bisector of $Q(a)$ is aligned with the positive y-axis, and b

lies to the right of this bisector; α and β are angles as in Fig. 2, with $\alpha \le \beta$. The cases where $|ab|$ is minimal (base case) or $ab \in Y_5$ are as discussed in the proof of Thm. 2. So let $c \in Q(a)$ and $d \in Q(b)$ be in S such that $\overrightarrow{ac} \in \overrightarrow{Y_5}$ and $\overrightarrow{bd} \in \overrightarrow{Y_5}$, and let $\phi = \angle cab$, and $\psi = \angle dba$ (see Fig. 3a).

Now, instead of applying Lem. 1 for the maximum value of ϕ (as in the proof of Thm. 2), we apply Lem. 1 only for values $\phi \le \overline{\theta}$ or $\psi \le \overline{\theta}$, for some threshold angle $\overline{\theta}$ (to be determined later). These cases yield a spanning ratio of $t \ge 1/(1 - 2\sin(\overline{\theta}/2))$. We handle the remaining cases differently, so for the remainder of the proof, we assume that $\phi > \overline{\theta}$ and $\psi > \overline{\theta}$. We compute an exact value of $\overline{\theta}$ shortly, but for now we only need that $\theta/2 < \overline{\theta} < 3\theta/4$. This implies that neither c nor d can lie to the right of ab, as this would make the corresponding angle smaller than $\theta/2$.

First consider the case where ac and bd intersect. In this case, instead of directly applying an inductive argument to either cb or da, we bound the distance cd and use induction to show that $|ac| + t|cd| + |db| \le t|ab|$. To derive this bound, consider the point c' such that $\angle c'ab = \overline{\theta}$ and $|ac'| = |ab|$ and the analogously defined point d' (see Fig. 3b). Let s be the intersection point between ac' and bd'. When ac and bd intersect, the distance $|cd|$ can be increased by rotating c towards b and d towards a. Since both ϕ and ψ must be larger than $\overline{\theta}$, the worst case occurs when $\phi = \psi = \overline{\theta}$, leaving c and d on the boundary of $\triangle c'd's$. As $c'd'$ is the longest side of this triangle, it follows that $|cd| < |c'd'|$. Using the fact that the triangles $\triangle c'd's$ and $\triangle abs$ are similar and isosceles, we can compute $|c'd'|$:

$$\begin{aligned}
|c'd'| &= 2|c's|\cos\overline{\theta} \\
&= 2(|ac'| - |as|)\cos\overline{\theta} \\
&= 2\left(|ab| - \frac{|ab|}{2\cos\overline{\theta}}\right)\cos\overline{\theta} \\
&= (2\cos\overline{\theta} - 1)|ab|
\end{aligned}$$

Recall that our aim is to use induction on cd to obtain a short path from a to b. We now compute the spanning ratio t required for the inequality $|ac| + t|cd| + |db| \le t|ab|$ to hold. By the inequality above, we have that $|ac| + t|cd| + |db| \le |ab| + t(2\cos\overline{\theta} - 1)|ab| + |ab|$. This latter term is bounded above by $t|ab|$ for any $t \ge 1/(1 - \cos\overline{\theta})$.

So far we derived two constraints on t and $\overline{\theta}$: $t \ge 1/(1 - 2\sin(\overline{\theta}/2))$ and $t \ge 1/(1 - \cos\overline{\theta})$. Because $\sin\overline{\theta}$ is increasing and $\cos\overline{\theta}$ is decreasing for all values of $\overline{\theta}$ under consideration, we minimize t by choosing $\overline{\theta}$ such that $1/(1 - 2\sin(\overline{\theta}/2)) = 1/(1 - \cos\overline{\theta})$. This yields $\overline{\theta} = \arccos\left(\sqrt{3} - 1\right) \approx 0.75$ and $t = 2 + \sqrt{3} \approx 3.74$.

Now consider what happens when one of ac or bd is "short", under some notion of short captured by the following lemma.

LEMMA 5. *Let $\triangle abc$ be a triangle with angle $\alpha = \angle cab$ and longest side ab. Let $\lambda > 1$ be a real constant. Then*

$$|ac| \le \frac{2\lambda^2\cos\alpha - 2\lambda}{\lambda^2 - 1}|ab| \quad \text{implies that} \quad |ac| + \lambda|bc| \le \lambda|ab|.$$

PROOF. The first inequality above implies $\lambda > 1/\cos\alpha$, otherwise $|ac|$ would be non-positive. By the law of cosines, $|bc| = \sqrt{|ac|^2 + |ab|^2 - 2|ab||ac|\cos\alpha}$. By substituting this in the inequality $|ac| + \lambda|bc| \le \lambda|ab|$, we see that it only holds if $|ac| \le \frac{2\lambda^2\cos\alpha - 2\lambda}{\lambda^2 - 1}|ab|$, as stated by the lemma. $\square$
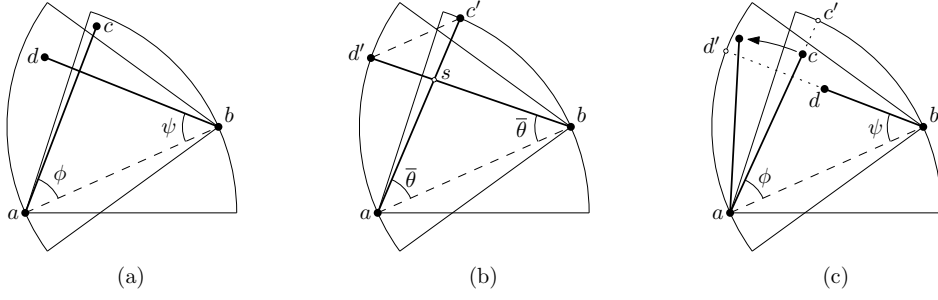
Figure 3: (a) Two vertices of Y_5 with their closest vertices. (b) The worst-case situation when ac and bd cross. (c) The rotation to maximize $|cd|$ when ac and bd do not cross.

The only case left to consider is when ac and bd are both long, but they do not intersect. In this case, we again seek to bound the distance $|cd|$. If we can show that $|cd| \leq (2\cos\overline{\theta} - 1)|ab|$, we can apply the same argument as for the intersecting case and we are done. Let c' be the point on the extension of ac with $|ac'| = |ab|$, and let d' be the analoguous point on the extension of bd (see Fig. 3c). If ac does not intersect bd', we can rotate d away from c by increasing ψ. Similarly, if bd does not intersect ac', we can rotate c away from d by increasing ϕ. Thus, the distance $|cd|$ is maximized when $\phi + \psi$ is maximal, which in our context happens when $\phi + \psi = 3\theta/2 = 3\pi/5$. Note that in most cases, rotating this far moves the corresponding vertex past the boundary of the cone. But since we are only trying to find an upper bound, this is not a problem.

Now let c'' be the point on the line through ac with $|ac''| = \frac{2t^2 \cos\phi - 2t}{t^2 - 1}|ab|$, and let d'' be the point on the line through bd with $|bd''| = \frac{2t^2 \cos\psi - 2t}{t^2 - 1}|ab|$. If c lies on ac'', Lem. 5 tells us that $|ac| + t|bc| \leq t|ab|$, which is exactly what we need. The only difficulty is that the location of c changed during the rotation. But since the rotation preserved $|ac|$ and only increased $|bc|$, the inequality must hold for the configuration before the rotation as well. The same argument applies for the case when d lies on bd''. The situation where c and d lie on $c''c'$ and $d''d'$, respectively, is handled by the following lemma.
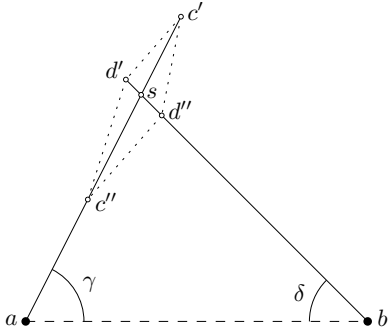

Figure 4: Illustration of Lem. 6

LEMMA 6. _Let $a, b, c, d \in S$. Let $\gamma = \angle cab$ and $\delta = \angle dba$ such that $\gamma > \overline{\theta}$, $\delta > \overline{\theta}$, and $\gamma + \delta = 3\pi/5$. Let $t = 2 + \sqrt{3}$. If $\frac{2t^2 \cos\gamma - 2t}{t^2 - 1}|ab| \leq |ac| \leq |ab|$ and $\frac{2t^2 \cos\delta - 2t}{t^2 - 1}|ab| \leq |bd| \leq |ab|$, then $|cd| \leq (2\cos\overline{\theta} - 1)|ab|$._

PROOF. Assume without loss of generality that $\gamma \geq \delta$ and $|ab| = 1$. Then $3\pi/10 \leq \gamma \leq 3\pi/5 - \overline{\theta}$ and $\overline{\theta} \leq \delta \leq 3\pi/10$. Let c' be the point on the extension of ac with $|ac'| = |ab|$, and let d' be the analoguous point on the extension of bd. Let s be the intersection of ac' and bd'. Let c'' be the point on the line through ac with $|ac''| = \frac{2t^2 \cos\gamma - 2t}{t^2 - 1}|ab|$, and let d'' be the point on the line through bd with $|bd''| = \frac{2t^2 \cos\delta - 2t}{t^2 - 1}|ab|$ (see Fig. 4). Let $c_1 = \frac{2t^2}{t^2 - 1}$ and $c_2 = \frac{1}{\sin(3\pi/5)}$. We derive (after some calculations)

$$\frac{d|ac''|}{d\gamma} = -c_1 \sin\gamma, \tag{1}$$

$$\frac{d|bd''|}{d\gamma} = c_1 \sin(3\pi/5 - \gamma), \tag{2}$$

$$\frac{d|as|}{d\gamma} = -c_2 \cos(3\pi/5 - \gamma), \tag{3}$$

$$\frac{d|bs|}{d\gamma} = c_2 \cos\gamma. \tag{4}$$

Let

$$x_1 = |as| - |ac''| = \frac{\sin\delta}{\sin(\gamma + \delta)} - \frac{2t^2 \cos\gamma - 2t}{t^2 - 1}, \tag{5}$$

$$x_2 = |ac'| - |as| = 1 - \frac{\sin\delta}{\sin(\gamma + \delta)}, \tag{6}$$

$$y_1 = |bs| - |bd''| = \frac{\sin\gamma}{\sin(\gamma + \delta)} - \frac{2t^2 \cos\delta - 2t}{t^2 - 1}, \tag{7}$$

$$y_2 = |bd'| - |bs| = 1 - \frac{\sin\gamma}{\sin(\gamma + \delta)}. \tag{8}$$

Note that the values of x_1 and y_1 could be negative if c'' or d'' lie past s. Substituting c_1, c_2, and (1) - (4) in the equalities above yields

$$\frac{dx_1}{d\gamma} = -c_2 \cos(3\pi/5 - \gamma) + c_1 \sin\gamma, \tag{9}$$

$$\frac{dx_2}{d\gamma} = c_2 \cos(3\pi/5 - \gamma), \tag{10}$$

$$\frac{dy_1}{d\gamma} = c_2 \cos\gamma - c_1 \sin(3\pi/5 - \gamma), \tag{11}$$

$$\frac{dy_2}{d\gamma} = -c_2 \cos\gamma. \tag{12}$$

Recall that $c_1 = \frac{2t^2}{t^2 - 1}$, $c_2 = \frac{1}{\sin(3\pi/5)}$, and $3\pi/10 \leq \gamma \leq 3\pi/5 - \overline{\theta}$. We verify the following:

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}\gamma^2} = -c_2 \sin(3\pi/5 - \gamma) + c_1 \cos\gamma$$
$$> -1.1\sin(3\pi/10) + 2.1\cos(3\pi/5 - \overline{\theta}) > 0,$$
$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}\gamma^2} = c_2 \sin(3\pi/5 - \gamma) > 0,$$
$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}\gamma^2} = -c_2 \sin\gamma + c_1 \cos(3\pi/5 - \gamma)$$
$$> -1.1\sin(3\pi/5 - \overline{\theta}) + 2.1\cos(3\pi/10) > 0,$$
$$\frac{\mathrm{d}^2 y_2}{\mathrm{d}\gamma^2} = c_2 \sin\gamma > 0.$$

Therefore, by plugging in $\gamma = 3\pi/10$ or $\gamma = 3\pi/5 - \overline{\theta}$ as the lower- or upper-bound of γ into (9) - (12), we can verify the following ranges:

$$-c_2 \cos(3\pi/10) + c_1 \sin(3\pi/10) \leq \frac{\mathrm{d}x_1}{\mathrm{d}\gamma},$$
$$-c_2 \cos\overline{\theta} + c_1 \sin(3\pi/5 - \overline{\theta}) \geq \frac{\mathrm{d}x_1}{\mathrm{d}\gamma},$$
$$c_2 \cos(3\pi/10) \leq \frac{\mathrm{d}x_2}{\mathrm{d}\gamma} \leq c_2 \cos\overline{\theta}, \quad (13)$$
$$c_2 \cos(3\pi/10) - c_1 \sin(3\pi/10) \leq \frac{\mathrm{d}y_1}{\mathrm{d}\gamma},$$
$$c_2 \cos(3\pi/5 - \overline{\theta}) - c_1 \sin\overline{\theta} \geq \frac{\mathrm{d}y_1}{\mathrm{d}\gamma},$$
$$-c_2 \cos(3\pi/10) \leq \frac{\mathrm{d}y_2}{\mathrm{d}\gamma},$$
$$-c_2 \cos(3\pi/5 - \overline{\theta}) \geq \frac{\mathrm{d}y_2}{\mathrm{d}\gamma}.$$

Specifically, we can verify that

$$\frac{\mathrm{d}x_1}{\mathrm{d}\gamma} \geq \max\left(\frac{\mathrm{d}x_2}{\mathrm{d}\gamma}, \left|\frac{\mathrm{d}y_1}{\mathrm{d}\gamma}\right|, \left|\frac{\mathrm{d}y_2}{\mathrm{d}\gamma}\right|\right), \quad (14)$$

which implies $\frac{\mathrm{d}(x_1 - x_2)}{\mathrm{d}\gamma} = \frac{\mathrm{d}x_1}{\mathrm{d}\gamma} - \frac{\mathrm{d}x_2}{\mathrm{d}\gamma} > 0$. By simply plugging in $\gamma = 3\pi/10$ into (5) and (6), we verify that $(x_1 - x_2) > 0$ when $\gamma = 3\pi/10$ and hence $x_1 > x_2$ for all $\gamma \in [3\pi/10, 3\pi/5 - \overline{\theta}]$. Similarly, we have $x_2 > 0$ when $\gamma = 3\pi/10$, and hence by (13), $x_2 > 0$ for all $\gamma \in [3\pi/10, 3\pi/5 - \overline{\theta}]$. These together yield $x_1 > x_2 > 0$. By the triangle inequality,

$$|c''d''| \leq |sc''| + |sd''| = |x_1| + |y_1| = x_1 + |y_1|,$$
$$|c''d'| \leq |sc''| + |sd'| = |x_1| + |y_2| = x_1 + |y_2|,$$
$$|c'd''| \leq |sc'| + |sd''| = |x_2| + |y_1| \leq x_1 + |y_1|,$$
$$|c'd'| \leq |sc'| + |sd'| = |x_2| + |y_2| \leq x_1 + |y_2|.$$

By (14),

$$\frac{\mathrm{d}(x_1 + |y_1|)}{\mathrm{d}\gamma} \geq \frac{\mathrm{d}(x_1)}{\mathrm{d}\gamma} - \left|\frac{\mathrm{d}(y_1)}{\mathrm{d}\gamma}\right| \geq 0,$$
$$\frac{\mathrm{d}(x_1 + |y_2|)}{\mathrm{d}\gamma} \geq \frac{\mathrm{d}(x_1)}{\mathrm{d}\gamma} - \left|\frac{\mathrm{d}(y_2)}{\mathrm{d}\gamma}\right| \geq 0.$$

By plugging in $\gamma = 3\pi/5 - \overline{\theta}$ into (5), (7), and (8), one can easily verify that $x_1 + |y_1| \leq 2\cos\overline{\theta} - 1$ and $x_1 + |y_2| \leq 2\cos\overline{\theta} - 1$ when γ is maximized. Therefore $\max(x_1 + |y_1|, x_1 + |y_2|) \leq 2\cos\overline{\theta} - 1$ for all $\gamma \in [3\pi/10, 3\pi/5 - \overline{\theta}]$, and hence $|cd| \leq \max(|c''d''|, |c''d'|, |c'd''|, |c'd'|) \leq 2\cos\overline{\theta} - 1$ as required. $\square$

This completes the proof for the upper bound. A proof establishing the following lower bound can be found in the full version of this paper [2].

THEOREM 7. Y_5 has spanning ratio at least 2.87.

3. SPANNING RATIO OF Y_6

In this section we fix $k = 6$ and show that, for any pair of points $a, b \in S$, $p(a, b) \leq 5.8|ab|$. We also establish a lower bound of 2 for the spanning ratio of Y_6. Our proof is inductive and it relies on two simple lemmas, which we introduce next.

Let $a, b \in S$ and let $\overrightarrow{ac} \in \overrightarrow{Y_6}$ be the edge from a within the cone that includes b. The next two lemmas will be relevant in the context where we seek to bound $p(a, b)$ by applying the induction hypothesis to $p(c, b)$. The basic geometry is illustrated in Fig. 5.
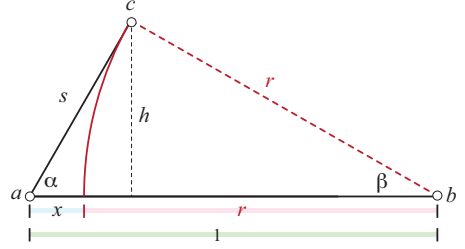

Figure 5: Notation for triangle $\triangle abc$. Here the dimensions have been normalized so that $|ab| = 1$.

LEMMA 8. [Triangle] Let $\triangle abc$ be labeled as in Fig. 5, with $|ac| \leq |ab|$, $|bc| < |ab|$, $x = |ab| - |bc|$ and $s = |ac|$. The ratio s/x is equal to some function t that depends on α and β:

$$\frac{s}{x} = t(\alpha, \beta) = \frac{\cos(\beta/2)}{\cos(\alpha + \beta/2)}. \quad (15)$$

PROOF. Normalize the triangle so that $|ab| = 1$; this does not alter the quantity we seek to compute, s/x. Let $|bc| = r$ to simplify notation. Then $x = 1 - r$ and $x \geq 0$ because $r = |bc| \leq |ab| = 1$. Note that each of the angles $\angle cab$ and $\angle cba$ is strictly less than $\pi/2$, because $|ac| \leq |ab|$ and $|bc| \leq |ab|$. Thus the projection of c onto ab is interior to the segment ab. Computing the altitude h of $\triangle abc$ in two ways yields

$$s \sin\alpha = r \sin\beta.$$

Also projections onto ab yield

$$s \cos\alpha + r \cos\beta = 1.$$

Solving these two equations simultaneously yields expressions for r and s as functions of α and β:

$$r = \frac{\sin\alpha}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}, \quad s = \frac{\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

Now we can compute $s/x = s/(1 - r)$ as a function of α and β. This simplifies to

$$\frac{s}{x} = \frac{\cos(\beta/2)}{\cos(\alpha + \beta/2)}$$

as claimed. $\square$

The following lemma derives an upper bound on the function $t(\alpha, \beta)$ from Lem. 8, which will be used in Thm. 10 to derive an optimal value for δ.

LEMMA 9. *Let $a, b, c \in S$ satisfy the conditions of Lem. 8, and let $t(\alpha, \beta)$ be as defined in (15). Let $\delta \in (0, \pi/3)$ be a fixed positive angle. If $\alpha \leq \pi/3 - \delta$, or $\beta \leq \pi/3 - \delta$, then*

$$t(\alpha, \beta) \leq t(\pi/3, \pi/3 - \delta) = \frac{\cos(\pi/6 - \delta/2)}{\sin(\delta/2)}.$$

PROOF. The derivative of $t(\alpha, \beta)$ with respect to α is

$$\frac{\partial t}{\partial \alpha} = \frac{\sin \alpha + \sin(\alpha + \beta)}{1 + \cos(2\alpha + \beta)} > 0.$$

This means that, for a fixed β value, $t(\alpha, \beta)$ reaches its maximum when α is maximum. Similarly, the derivative of $t(\alpha, \beta)$ with respect to β is

$$\frac{\partial t}{\partial \beta} = \frac{\sin \alpha}{2\cos(\alpha + \beta/2)^2} > 0.$$

So for a fixed value α value, $t(\alpha, \beta)$ reaches its maximum when β is maximum. Because $|ac| \leq |ab|$, $\beta \leq \angle acb$. The sum of these two angles is $\pi - \alpha$, therefore $\beta \leq \pi/2 - \alpha/2$. This along with the derivations above implies that, for a fixed value $\alpha \leq \pi/3 - \delta$, $t(\alpha, \beta) \leq t(\alpha, \pi/2 - \alpha/2) \leq t(\pi/3 - \delta, \pi/3 + \delta/2)$ (we substituted $\alpha = \pi/3 - \delta$ in this latter inequality). Next we evaluate

$$\frac{t(\pi/3, \pi/3 - \delta)}{t(\pi/3 - \delta, \pi/3 + \delta/2)} = \frac{\cos(\pi/6 - \delta/2)\sin(3\delta/2)}{\cos(\pi/6 + \delta/2)\sin(\delta/2)} > 1.$$

It follows that $t(\pi/3, \pi/3 - \delta)$ is maximal. $\square$

We are now ready to prove the main result of this section.

THEOREM 10. *Y_6 has spanning ratio at most 5.8.*

This result follows from the following lemma, with the variable δ substituted by the quantity $\delta_0 = 0.324$ that minimizes $t(\delta)$. (It can be easily verified that $t(\delta) \geq t(0.324)$ and $t(0.324) < 5.8$.)

LEMMA 11. *Let $\delta \in (0, \pi/9)$ be a strictly positive real value. The graph Y_6 has spanning ratio bounded above by*

$$t = t(\delta) = \max\left\{ \frac{\cos(\pi/6 - \delta/2)}{\sin(\delta/2)}, \frac{2}{1 - \frac{\sin(2\delta)}{\sin(\pi/6 + 2\delta)}} \right\}. \quad (16)$$

PROOF. The proof is by induction on the pairwise distance between pairs of points $a, b \in S$. Without loss of generality let $b \in Q_0(a)$.

Base case.

We show that, if $|ab|$ is minimal, then $\overrightarrow{ab} \in \overrightarrow{Y_6}$ and so $p(a, b) = |ab|$. If $\overrightarrow{ab} \in \overrightarrow{Y_6}$, then the lemma holds. So assume that $\overrightarrow{ab} \notin \overrightarrow{Y_6}$; we will derive a contradiction. Because $\overrightarrow{ab} \notin \overrightarrow{Y_6}$, there must be another point $c \in Q_0(a)$ such that $\overrightarrow{ac} \in \overrightarrow{Y_6}$ and $|ac| = |ab|$. Let α_1 and α_2 be the angles that ab and ac make with the horizontal respectively. Because both $\alpha_1, \alpha_2 \in [0, \pi/3)$, necessarily $|\alpha_1 - \alpha_2| < \pi/3$. Thus $|bc| < |ab| = |ac|$, contradicting the assumption that $|ab|$ is minimal. So in fact it must be that $\overrightarrow{ab} \in \overrightarrow{Y_6}$, and the lemma is established.

Main idea of the inductive step.

It has already been established that Y_7 is a spanner [4]; the sector angles for Y_7 are $2\pi/7$. The main idea of our inductive proof is to partition the $\pi/3$-sectors of Y_6 into peripheral cones of angle δ, for some fixed $\delta \in (0, \pi/9)$, leaving a central sector of angle $\pi/3 - 2\delta$. (The δ-cones are the shaded regions in Fig. 6.)

When an edge of Y_6 falls inside the central sector, induction will apply, because an edge within the central sector makes definite progress toward the goal in that sector (as it does in Y_7), ensuring that the remaining distance to be covered is strictly smaller than the original. This idea is captured by the flowing lemma.

LEMMA 12. *[Induction Step] Let $a, b, c \in S$ such that b and c lie in the same cone with apex a, and $\overrightarrow{ac} \in Y_6$. Let $\alpha = \angle cab$ and $\beta = \angle cba$. If either $\alpha < \pi/3 - \delta$ or $\beta < \pi/3 - \delta$, then we may use induction on $p(c, b)$ to conclude that $p(a, b) \leq t|ab|$.*

PROOF. This configuration is depicted in Fig. 5. Because $\overrightarrow{ac} \in Y_6$ and b and c lie in the same cone with apex a, we have that $|ac| \leq |ab|$. Because at least one of α or β is strictly smaller than $\pi/3$, we have that $|cb| < |ab|$. Thus the conditions of Lem. 8 are satisfied, so we can use Lem. 8 to bound $|ac|$ in terms of $x = |ab| - |bc|$: since $|ac|/x < t$, $|ac| < tx$. Because $|cb| < |ab|$, we may apply induction to bound $p(c, b)$: $p(c, b) \leq t|cb|$. Hence

$$p(a, b) \leq |ac| + p(c, b) \leq tx + t|cb| = t(x + |cb|) = t|ab|.$$

$\square$

We will henceforth use the symbol $\boxed{Induct}$ as shorthand for applying Lem. 12 to a triangle equivalent to that in Fig. 5.

Lem. 12 leaves out Y_6 edges falling within the δ-cones, that could conceivably *not* make progress toward the goal. For example, following one edge of an equilateral triangle leaves one exactly as far away from the other corner as at the start. However, we will see that when all relevant edges of Y_6 fall with the δ-cones near $\pi/3$, the restricted geometric structure ensures that progress toward the goal is indeed made, and again induction applies.

Inductive step.

The inductive step proof first handles the cases where edges of Y_6 directed from a or from b fall in the central portion of the relevant sectors, and so satisfy Lem. 9, and so Lem. 12 applies.

Recall that $b \in Q_0(a)$ by our assumption. If $\overrightarrow{ab} \in \overrightarrow{Y_6}$, then $p(a, b) = |ab|$ and we are finished. Assuming otherwise, there must be a point $c \in Q_0(a)$ such that $\overrightarrow{ac} \in \overrightarrow{Y_6}$ and $|ac| \leq |ab|$. For the remainder of the proof, we are in this situation, with $ac \in Y_6$ and $|ac| \leq |ab|$. The proof now partitions into three parts: (1) when only $Q_0(a)$ is relevant and leads to $\boxed{Induct}$; (2) when $Q_2(b)$ leads to $\boxed{Induct}$; (3) when we fall into a special situation, for which induction also applies, but for different reasons.

(1) The $Q_0(a)$ sector.

Consider $\triangle abc$ as previously illustrated in Fig. 5. If either b or c is not in one of the δ-cones of $Q_0(a)$, then $\alpha = \angle bac < \pi/3 - \delta$: $\boxed{Induct.}$
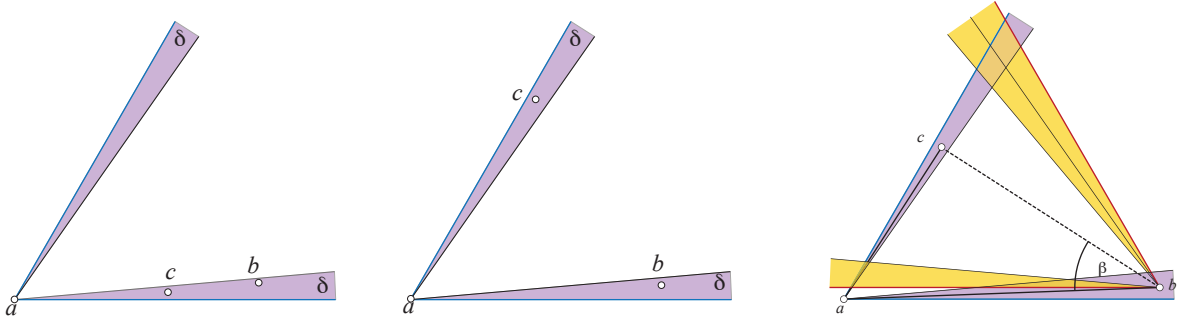
Figure 6: The δ-cones are shaded. (a) b and c in the same δ-cone. (b) b and c in different δ-cones. (c) $Q_0(a)$ and $Q_2(b)$. Here β is "small".

Now assume that both b and c lie in δ-cones of $Q_0(a)$. If they both lie within the same δ-cone (Fig. 6a), then again α is small: Induct. So without loss of generality let b lie in the lower δ-cone, and c in the upper δ-cone of $Q_0(a)$; see Fig. 6b. We cannot apply induction in this situation because the ratio s/x in Lem. 8 has no upper bound.

(2) The $Q_2(b)$ sector.

Now we consider $Q_2(b)$, the sector with apex at b aiming to the left of b, and assume that $c \in Q_2(b)$. Refer to Fig. 6c. The case $c \notin Q_2(b)$ will be discussed later (special situation).

Because b may subtend an angle as large as δ at a with the horizontal, the "upper 2δ-cone" of $Q_2(b)$ becomes the relevant region. If c is not in the upper 2δ-cone of $Q_2(b)$ (as depicted in Fig. 6c), then $\triangle abc$ satisfies Lem. 9 with $\beta < \pi/3 - \delta$: Induct. Note that this conclusion follows even if c is in the small region outside of and below $Q_2(b)$: the angle β at b is then very small.

Assume now that c is in the upper 2δ-cone of $Q_2(b)$. Let $d \in Q_2(b)$ be the point such that $\overrightarrow{bd} \in \overrightarrow{Y_6}$. We now consider possible locations for d. If $d = c$, then $p(a, b) \leq |ac| + |cb| \leq 2|ab|$, and we are finished. So assume henceforth that d is distinct from c.

If d is not in the upper δ-cone of $Q_0(a)$ (Fig. 7a), then $\triangle abd$ satisfies Lem. 9 with the roles of a and b reversed: bd takes a step toward a, with the angle at a satisfying $\angle bad < \pi/3 - \delta$: Induct.

If d is not in the upper 2δ-cone of $Q_2(b)$ (Fig. 7b), then $\triangle abd$ satisfies Lem. 9 again with the roles of a and b reversed and this time the angle at b bounded away from $\pi/3$, $\angle abd < \pi/3 - \delta$: Induct.

Assume now that d is in the intersection region between the upper δ-cone of $Q_0(a)$ and the upper 2δ-cone of $Q_2(b)$. Recall that we are in the situation where c lies in the same region, so it is close to d. See Fig. 7c. This suggests the strategy of following ac and db, connected by $p(c, d)$. We show that in fact $|cd| < |ab|$, so the inductive hypothesis can be applied to $p(c, d)$. More precisely, we show the following result.

LEMMA 13. Let $a, b, c, d \in S$ be as in Fig. 7c, with $\overrightarrow{bd} \in Y_6$, $b, c \in Q_0(a)$ and $c, d \in Q_2(b)$. If both c and d lie above the lower rays bounding the upper 2δ-cones of $Q_0(a)$ and $Q_2(b)$, then for any $0 \leq \delta \leq \pi/9$,

$$|cd| \leq \frac{\sin(2\delta)}{\sin(\pi/6 + 2\delta)}|ab|. \qquad (17)$$

Note that c lies in the intersection region between the upper 2δ-cones of $Q_0(a)$ and $Q_2(b)$, because $c \in Q_0(a) \cap Q_2(b)$ (by the statement of the lemma). However, Lem. 13 does not restrict the location of d to the same region. Indeed, d may lie either below or above the upper ray bounding $Q_0(a)$, as long as it satisfies the condition $|bd| \leq |bc|$. (This condition must hold because c, d are in the same sector $Q_2(b)$, and $\overrightarrow{bd} \in Y_6$.) To keep the flow of our main proof uninterrupted, we defer a proof of Lem. 13 to Section 4.1.

By Lem. 13 we have $|cd| < |ab|$. Thus we can use the induction hypothesis to show that $p(c, d) \leq t|cd|$. We know that $|ac| \leq |ab|$ because both b and c are in $Q_0(a)$ and $\overrightarrow{ac} \in \overrightarrow{Y_6}$. We also know that $|bd| \leq |bc|$ because both c and d are in $Q_2(b)$ and $\overrightarrow{bd} \in \overrightarrow{Y_6}$. Let u and i be the upper and lower intersection points between the rays bounding $Q_2(b)$ and the upper ray of $Q_0(a)$, as in Fig. 7c. Note that $\triangle bui$ is equilateral, and because c lies in this triangle, we have $|bc| \leq |bu| = |bi| \leq |ab|$. It follows that $|bd| \leq |ab|$. So in this situation (illustrated in Fig. 7c), we have:

$$
\begin{aligned}
p(a, b) &\leq |ac| + p(c, d) + |bd| \\
&\leq 2|ab| + p(c, d) \\
&\leq 2|ab| + t|cd| \\
&\leq 2|ab| + t\frac{\sin(2\delta)}{\sin(\pi/6 + 2\delta)}|ab| \\
&\leq t|ab|.
\end{aligned}
$$

Here we have applied Lem. 13 to bound $|cd|$. Note that the latter inequality above is true for the value of t from (16).

(3) Special situation.

The only case left to discuss is the one in which c lies in the upper δ-cone of $Q_0(a)$ and to the right of the upper ray of $Q_2(b)$. This situation is depicted in Fig. 8. Next consider $Q_4(c)$. Because $b \in Q_4(c)$, there exists $\overrightarrow{cz} \in Y_6$, with $z \in Q_4(c)$ and $|cz| \leq |cb|$. Clearly $z \in Q_0(a) \cup Q_5(a)$. Note that the disk sector $D_0(a, |ac|) \subset Q_0(a)$ with center a and radius $|ac|$ must be empty, because $\overrightarrow{ac} \in Y_6$.

Case 3(a).

If $z \in Q_0(a)$, then z lies in the lower δ-cone of $Q_0(a)$ and to the right of $D_0(a, |ac|)$, close to b. See Fig. 8a. In this case we show that the quantity on the right side of inequality (17) is a loose upper bound on $|bz|$, and that similar inductive arguments hold here as well. Let the circumference of $D_0(a, |ac|)$ intersect the right ray of $Q_4(c)$ and the lower ray of $Q_0(a)$ at points $z' \neq c$ and b', respectively. Refer to
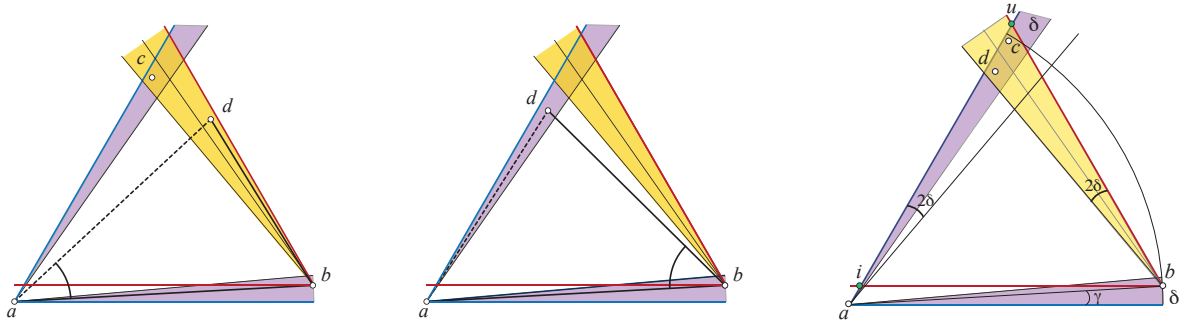
Figure 7: (a) d not in the upper δ-cone of $Q_0(a)$: $\angle bad$ is small. (b) d not in the upper 2δ-cone of $Q_2(b)$: $\angle abd$ is small. (c) Lem. 13: $|cd| < |ab|$.
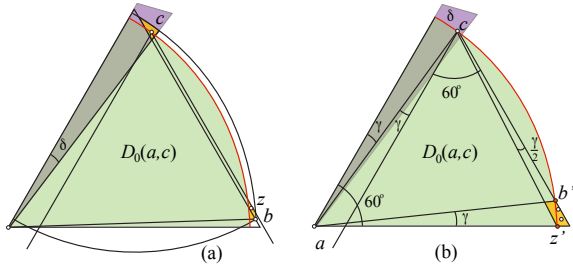

Figure 8: Case $c \notin Q_2(b)$, $z \in Q_0(a)$: Lem. 13 applies on bd.

Fig. 8b. Let $\gamma \leq \delta$ be the angle formed by ac with the upper ray of $Q_0(a)$. Then $\angle z'ab' = \gamma$ and $\angle z'cb' = \gamma/2$. This implies that both b' and z' lie in the intersection region between the lower δ-cone of $Q_0(a)$ and the right $\delta/2$-cone of $Q_4(c)$. Thus $a, b, c, z \in S$ satisfy the conditions of Lem. 13, with the roles of b and c reversed: $|bz| \leq \sin(2\delta)/\sin(\pi/6 + 2\delta) \cdot |ac|$.

Arguments similar to the ones used for the case depicted in Fig. 10a show that $|cz| \leq |ac|$. This along with $|ac| \leq |ab|$ (because $\overrightarrow{ac} \in Y_6$) and the above inequality imply

$$p(a,b) \leq |ac| + |cz| + p(z,b) \leq 2|ab| + t|bz| \leq t|ab|$$

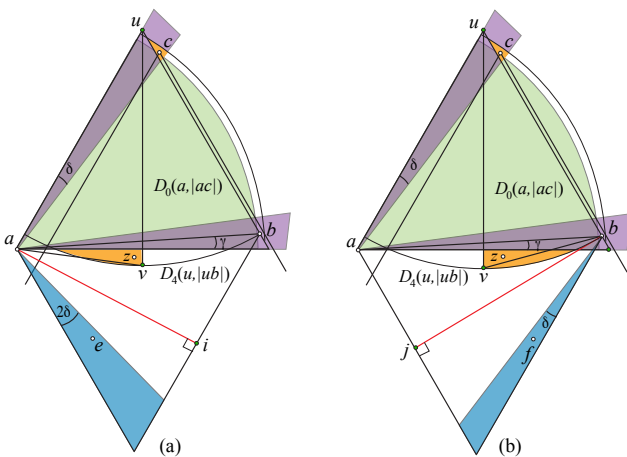for any t satisfying the conditions stated by this lemma.

Figure 9: Case $c \notin Q_2(b)$ and $z \in Q_5(a)$ (a) z left of uv (b) z right of uv.

Case 3(b).

Assume now that $z \notin Q_0(a)$. Then $z \in Q_5(a)$, as depicted in Fig. 9. In this case z lies in the disk sector $D_4(c, |cb|)$ (because $|cz| \leq |cb|$) and below the horizontal through a (because $D_0(a, |ac|)$ is empty). This implies that there exists $\overrightarrow{ae} \in Y_6$, with $e \in Q_5(a)$ and $|ae| \leq |az|$. Similarly, there exists $\overrightarrow{bf} \in Y_6$, with $f \in Q_3(b)$ and $|bf| \leq |bz|$. If e lies above the lower 2δ-cone of $Q_5(a)$, then $\angle bae \leq \pi/3 - \delta$, which leads to $\boxed{Induct}$ and settles this case. Similarly, if f lies above the lower δ-cone of $Q_3(b)$, then $\angle abf \leq \pi/3 - \delta$, which again leads to $\boxed{Induct}$. Otherwise, we show that the following lemma holds.

LEMMA 14. *Let $a, b, c, z \in S$ be in the configuration depicted in Fig. 9, with $\overrightarrow{ac}, \overrightarrow{cz} \in Y_6$. Let $\overrightarrow{ae}, \overrightarrow{bf} \in Y_6$, with e in the lower 2δ-cone of $Q_5(a)$ and f in the lower δ-cone of $Q_3(b)$. Then at least one of the following is true: (a) $e \in Q_3(b)$, or (b) $f \in Q_5(a)$.*

We defer a proof of Lem. 14 to Section 4.2.

Lem. 14 guarantees that, if condition (a) holds, then ae may not cross the lower ray bounding $Q_3(b)$. This case reduces to one of the cases depicted in Figs. 7 and 10, with e playing the role of c and the path passing under ab rather than above. Because ae does not cross the lower ray bounding $Q_3(b)$, the special situation depicted in Fig. 8 (with e playing the role of c) may not occur in this case. Similarly, condition (b) from Lem. 14 reduces to one of the cases depicted in Figs. 7 and 10, with the roles of a and b reversed and with f playing the role of c; the special situation depicted in Fig. 8 (with bf playing the role of ac) may not occur in this case. Having exhausted all cases, we conclude the proof. □

A proof establishing the following lower bound can be found in the full version of the paper [2].

THEOREM 15. *Y_6 has spanning ratio at least 2.*

4. DEFERRED PROOFS

4.1 Proof of Lemma 13

LEMMA 13. *Let $a, b, c, d \in S$ be as in Fig. 10a, with $\overrightarrow{bd} \in Y_6$, $b, c \in Q_0(a)$ and $c, d \in Q_2(b)$. If both c and d lie above the lower rays bounding the upper 2δ-cones of $Q_0(a)$ and $Q_2(b)$, then for any $0 \leq \delta \leq \pi/9$,*

$$|cd| \leq \frac{\sin(2\delta)}{\sin(\pi/6 + 2\delta)}|ab|$$

PROOF. Let u and v be the top and bottom points of the intersection quadrilateral R between the upper 2δ-cones of $Q_0(a)$ and $Q_2(b)$. See Fig. 10. Then $c \in R$. For any $\delta \leq \pi/9$, the angles opposite to the diagonal uv of R are bounded below by $5\pi/9$, therefore uv is the diameter of R.
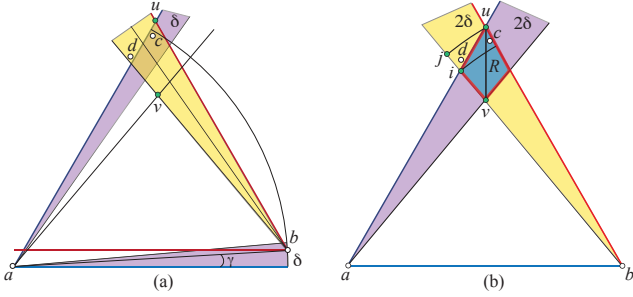

Figure 10: Lem. 13: (a) uv is minimum when $\gamma = 0$. (b) $|cd| \leq |uv|$.

Assume first that $d \in R$ as well. In this case, the quantity $|cd|$ is bounded above by the length $|uv|$ of the diameter of R. Let γ be the angle formed by ab with the horizontal. We show that $|uv|$ is maximized when $\gamma = 0$. Set a coordinate system with the origin at a. Scale the point set S so that $|ab| = 1$. Then the coordinates of b are $(\cos\gamma, \sin\gamma)$. The point u is at the intersection of the two lines passing through a and b with slopes $\tan\pi/3$ and $-\tan\pi/3$ respectively, given by $y = \sqrt{3}x$ and $y = -\sqrt{3}(x - \cos\gamma) + \sin\gamma$. Solving for x and y gives the coordinates of u

$$x_u = \frac{\sqrt{3}\cos\gamma + \sin\gamma}{2\sqrt{3}}, \qquad y_u = \frac{\sqrt{3}\cos\gamma + \sin\gamma}{2}.$$

Similarly, the point v is at the intersection of two lines given by $y = \tan(\pi/3 - 2\delta)x$ and $y = -\tan(\pi/3 - 2\delta)(x - \cos\gamma) + \sin\gamma$. Solving for x and y gives the coordinates of v

$$x_v = \frac{\tan(\pi/3 - 2\delta)\cos\gamma + \sin\gamma}{2\tan(\pi/3 - 2\delta)},$$

$$y_v = \frac{\tan(\pi/3 - 2\delta)\cos\gamma + \sin\gamma}{2}.$$

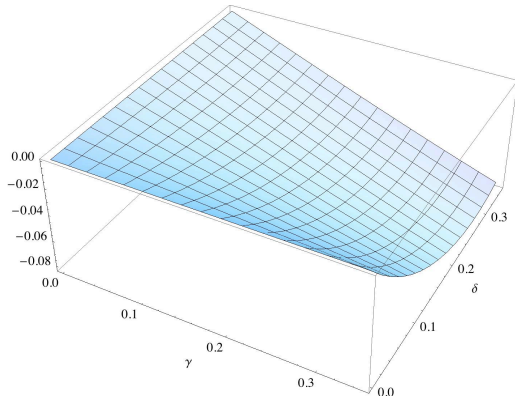We can now compute $|uv| = \sqrt{(x_u - x_v)^2 + (y_u - y_v)^2}$ as

Figure 11: Lem. 13. The derivative of $|uv|$ with respect to γ, for $\gamma, \delta \in [0, \pi/9]$.

a function of γ and δ. The derivative of this function with respect to γ is represented as a graph in Fig. 11 for $\gamma, \delta \in [0, \pi/9]$. Note that this function is negative on the given interval, therefore $|uv|$ increases as γ decreases. Thus $|uv|$ is maximum when $\gamma = 0$. We now set $\gamma = 0$ and compute $|uv| \leq \sqrt{3}/2 - \cot(2\delta + \pi/6)/2 = \sin(2\delta)/\sin(2\delta + \pi/6)$ as claimed.

Assume now that $d \notin R$, so d lies above the upper ray bounding $Q_0(a)$. Let i be the intersection point between the upper ray bounding $Q_0(a)$ and the lower ray bounding the upper 2δ-cone of $Q_2(b)$. Then c must lie outside the disk $D_2(b, |bi|)$, because d lies outside this disk (by assumption) and $|bd| \leq |bc|$ (because $\overrightarrow{bd} \in Y_6$). Refer to Fig. 10b. Let j be the intersection point between the lower ray bounding the upper 2δ-cone of $Q_2(b)$ and the circumference of $D_2(b, |bu|)$. Then both c and d lie in the strip delimited by $D_2(b, |bi|)$, $D_2(b, |bj|)$ and the two rays bounding the upper 2δ-cone of $Q_2(b)$. Thus cd is no greater than the diameter of this strip, which we show to be no greater than the diameter of R. For this, it suffices to show that $\max\{|ui|, |uj|, |ij|\} \leq |uv|$.

Because ui is an edge of R, $|ui|$ is clearly no greater than the diameter $|uv|$ of R. Next we show that $|uj| \leq |uv|$. From the isosceles triangle $\triangle buj$ we derive $\angle ujv = \pi/2 - \delta$. Angle $\angle uvj$ is exterior to $\triangle uvb$, therefore $\angle uvj = \angle vub + 2\delta \leq \pi/6 + 2\delta$ (note that $\angle vub = \pi/6$ when ab is horizontal, otherwise $\angle vub < \pi/6$). It follows that $\angle ujv \leq \angle uvj$ for any $\delta \leq \pi/9$. This along with the law of sines applied to $\triangle ujv$ yields $|uj| \leq |uv|$.

It remains to show that $|ij| < |uv|$. We will in fact show that $|ij| < |uj|$, which along with the conclusion above that $|uj| \leq |uv|$, yields $|ij| < |uv|$. Angle $\angle uij$ is exterior to $\triangle uib$, therefore $\pi/3 \leq \angle uij \leq \pi/3 + 2\delta$. Earlier we showed that $\angle ujv = \pi/2 - \delta \geq 7\pi/18$, for any $\delta \leq \pi/9$. It follows that $\angle iuj \leq \pi - (7\pi/18 + \pi/3) = 5\pi/18$ is the smallest angle of $\triangle uij$, therefore $|ij| < |uv|$. This completes the proof. $\square$

4.2 Proof of Lemma 14

LEMMA 14. *Let* $a, b, c, z \in S$ *be in the configuration depicted in Fig. 9, with* $\overrightarrow{ac}, \overrightarrow{cz} \in Y_6$. *Let* $\overrightarrow{ae}, \overrightarrow{bf} \in Y_6$, *with* e *in the lower* 2δ-*cone of* $Q_5(a)$ *and* f *in the lower* δ-*cone of* $Q_3(b)$. *Then at least one of the following is true:*

(a) $e \in Q_3(b)$

(b) $f \in Q_5(a)$

PROOF. We define four intersection points u, v, i and j as follows: u is at the intersection between the top rays of $Q_0(a)$ and $Q_2(b)$; v is at the intersection between the bisector of $\angle aub$ and the boundary of the disk sector $D_4(u, |ub|)$; i is the foot of the perpendicular from a on the lower ray of $Q_3(b)$; and j is the foot of the perpendicular from b on the lower ray of $Q_5(a)$. Refer to Fig. 9.

Note that $|ae| \leq |ai|$ implies condition (a), and $|bf| \leq |bj|$ implies condition (b). We show that the first holds if z lies to the left of or on uv, and the latter holds if z lies to the right of or on uv (and so at least one of the two conditions holds). We first show that $z \in D_4(u, |ub|)$. This follows immediately from the inequality $|uz| + |cb| < |cz| + |ub|$ (which can be derived using the triangle inequality twice on the triangles induced by the diagonals of $ucbz$), and the fact that $|cz| \leq |cb|$ (because $\overrightarrow{cz} \in Y_6$). It follows that $|uz| < |ub|$, therefore $z \in D_4(u, |ub|)$.

Condition (a).

Assume that z lies to the left of uv (as in Fig. 9a). Because $z \in D_4(u, |ub|)$ is below the horizontal through a, $\angle azv$ is obtuse and therefore $|az| \le |av|$ (equality holds when z coincides with v). Also $|ae| \le |az|$, because z and e are in the same sector $Q_5(a)$ and $\overrightarrow{ae} \in Y_6$. It follows that $|ae| \le |av|$. We now show that $|av| \le |ai|$, which implies $|ae| \le |ai|$, thus settling this case.

Let $\gamma \in [0, \delta]$ be the angle formed by ab with the horizontal through a. Then $\angle abi = \pi/3 - \gamma$ and $|ai| = |ab| \sin(\pi/3 - \gamma)$. The law of sines applied to $\triangle uav$ tells us that

$$\frac{|av|}{\sin \pi/6} = \frac{|ua|}{\sin \angle uva} = \frac{|uv|}{\sin \angle uav}.$$

Note that $|uv| = |ub| \le |ua|$, because v lies on the circumference of $D(u, |ub|)$ and a lies outside of this disk. This along with the latter equality above yields $\angle uav \le \angle uva$. The sum of these two angles is $5\pi/6$ (recall that uv is the bisector of $\angle aub$), therefore $\angle uva \ge 5\pi/12$. Also note that $\angle uva < \pi/2$, because v lies strictly below the horizontal through a (otherwise d may not exist). It follows that $\sin \angle uva \ge \sin 5\pi/12$. Substituting this in the equality above yields $|av| \le |ua| \sin \pi/6 / \sin 5\pi/12$. The law of sines applied to triangle $\triangle abu$ yields $|au| = |ab| \sin(\pi/3 + \gamma) / \sin \pi/3$, which substituted in the previous equality yields

$$|av| \le |ab| \frac{\sin(\pi/3 + \gamma) \sin \pi/6}{\sin \pi/3 \sin 5\pi/12}.$$

Thus the inequality $|av| \le |ai|$ holds for any γ satisfying

$$\frac{\sin(\pi/3 + \gamma) \sin \pi/6}{\sin \pi/3 \sin 5\pi/12} \le \sin(\pi/3 - \gamma).$$

It can be easily verified that this inequality holds for any $\gamma \le \delta \le 23\pi/180$, and in particular for the δ values restricted by Lem. 11.

Condition (b).

Assume now that z lies to the right of uv (as in Fig. 9b). In this case $|bf| \le |bz| \le |bv|$. We now show that $|bv| \le |bj|$, which implies $|bf| \le |bj|$, thus settling this case. From the right triangle $\triangle baj$ with angle $\angle baj = \pi/3 + \gamma$, we derive $|bj| = |ab| \sin(\pi/3 + \gamma)$. Next we derive an upper bound on $|bv|$. From the isosceles triangle $\triangle vub$, having angle $\angle vub = \pi/6$, we derive $|bv| = 2|bu| \sin \pi/12$. The law of sines applied to triangle $\triangle uab$ gives us $|ub| = |ab| \sin(\pi/3 - \gamma) / \sin \pi/3$, which substituted in the previous equality yields $|bv| = 2|ab| \sin(\pi/3 - \gamma) \sin \pi/12 / \sin \pi/3$. Thus the inequality $|bv| \le |bj|$ holds for any γ value satisfying

$$\frac{2 \sin(\pi/3 - \gamma) \sin \pi/12}{\sin \pi/3} \le \sin(\pi/3 + \gamma).$$

It can be verified that this inequality holds for any $\gamma \le \delta \le \pi/3$, and in particular for the δ values restricted by Lem. 11. $\square$

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