On Edge-Disjoint Empty Triangles of Point Sets

Javier Cano, Luis F. Barba, Toshinori Sakai, and Jorge Urrutia

Abstract Let P be a set of points in the plane in general position. Any three points $x, y, z \in P$ determine a triangle $\Delta(x, y, z)$ of the plane. We say that $\Delta(x, y, z)$ is empty if its interior contains no element of P. In this chapter, we study the following problems: What is the size of the largest family of edge-disjoint triangles of a point set? How many triangulations of P are needed to cover all the empty triangles of P? We also study the following problem: What is the largest number of edge-disjoint triangles of P containing a point q of the plane in their interior? We establish upper and lower bounds for these problems.

1 Introduction

Let P be a set of n points in the plane in general position. A geometric graph on P is a graph G whose vertices are the elements of P, two of which are adjacent if they are joined by a straight-line segment. We say that G is a plane if it has no edges that cross each other. A triangle of G consists of three points $x, y, z \in P$ such that xy, yz, and zx are edges of G; we will denote it as $\Delta(x, y, z)$. If, in addition, $\Delta(x, y, z)$ contains no elements of P in its interior, we say that it is empty.

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J. Pach (ed.), *Thirty Essays on Geometric Graph Theory*,
DOI 10.1007/978-1-4614-0110-0_7,

In a similar way, we say that if $x,y,z \in P$, then $\Delta(x,y,z)$ is a *triangle* of P, and that xy, yz, and zx are the edges of $\Delta(x,y,z)$. If $\Delta(x,y,z)$ is empty, it is called a 3-*hole* of P. A 3-hole of P can be thought of as an empty triangle of the complete geometric graph $\mathcal{K}_P$ on P. We remark that $\Delta(x,y,z)$ will denote a triangle of a geometric graph and also a triangle of a point set.

A well-known result in graph theory says that for $n = 6k+1$ or $n = 6k+3$, the edges of the complete graph K_n on n vertices can be decomposed into a set of $\binom{n}{2}/3$ edge-disjoint triangles. These decompositions are known as Steiner triple systems [23]; see also Kirkman's schoolgirl problem [17, 22]. In this chapter, we address some variants of that problem, but for geometric graphs.

Given a point set P, let $\delta(P)$ be the size of the largest set of edge-disjoint empty triangles of P. It is easy to see that for point sets in convex position with $n = 6k+1$ or $n = 6k+3$ elements, $\delta(P) = \binom{n}{2}/3$. Indeed, any triangle of P is empty, and the problem is the same as that of decomposing the edges of the complete geometric graph $\mathcal{K}(P)$ on P into edge-disjoint triangles. On the other hand, we prove that for some point sets, namely Horton point sets, $\delta(P)$ is $O(n\log n)$.

We then study the problem of covering the empty triangles of point sets with as few triangulations of P as possible. For point sets in convex position, we prove that we need essentially $\binom{n}{3}/4$ triangulations; our bound is tight. We also show that there are point sets P for which $O(n\log n)$ triangulations are sufficient to cover all the empty triangles of P for a given point set P.

Finally, we consider the problem of finding a point q not in P contained in the interior of many edge-disjoint triangles of P. We prove that for any point set, there is a point $q \notin P$ contained in at least $n^2/12$ edge-disjoint triangles. Furthermore, any point in the plane, not in P, is contained in at most $n^2/9$ edge-disjoint triangles of P, and this bound is sharp. In particular, we show that this bound is attained when P is the set of vertices of a regular polygon.

1.1 Preliminary Work

The study of counting and finding k-holes in point sets has been an active area of research since Erdős and Szekeres [11, 12] asked about the existence of k-holes in planar point sets. It is known that any point set with at least 10 points contains 5-holes; e.g., see [14]. Horton [15] proved that for $k \geq 7$, there are point sets containing no k-holes. The question of the existence of 6-holes remained open for many years, but recently Nicolás [19] proved that any point set with sufficiently many points contains a 6-hole. A second proof of this result was subsequently given by Gerken [13].

The study of properties of the set of triangles generated by point sets on the plane has been of interest for many years. Let $f_k(n)$ be the minimum number of k-holes that a point set has. Katchalski and Meir [16] proved that $\binom{n}{2} \leq f_3(n) \leq cn^2$ for some $c < 200$; see also Purdy [21]. Their lower bounds were improved by Dehnhardt [9] to $n^2 - 5n + 10 \leq f_3(n)$. He also proved that $\binom{n-3}{2} + 6 \leq f_4(n)$. Point sets with few k-holes for $3 \leq k \leq 6$ were obtained by Bárány and Valtr [2]. The interested reader can read [18] for a more accurate picture of the developments in this area of research.

Chromatic variants of the Erdős–Szekeres problem have recently been studied by Devillers, Hurtado, Károly, and Seara [10]. They proved among other results that any bichromatic point set contains at least $\frac{n}{4} - 2$ compatible monochromatic empty triangles. Aichholzer et al. [1] proved that any bichromatic point set always contains $\Omega(n^{5/4})$ empty monochromatic triangles; this bound was improved by Pach and Tóth [20] to $\Omega(n^{4/3})$.

2 Sets of Edge-Disjoint Empty Triangles in Point Sets

Let P be a set of points in the plane, and let $\delta(P)$ be the size of the largest set of edge-disjoint empty triangles of the complete graph $\mathcal{K}(P)$ on P. In this section we study the following problem:

Problem 1. How small can $\delta(P)$ be?

We show that if P is a Horton set, then $\delta(P)$ is $O(n \log n)$. By Kirkman's result, for points in convex position with $n = 6k+1$ and $n = 6k+3$, $\delta(P)$ is $\frac{\binom{n}{3}}{3}$.

For any integer $k \geq 1$, Horton [15] recursively constructed a family of point sets H_k of size 2^k as follows:

(a) $H_1 = \{(0,0), (1,0)\}$.
(b) H_k consists of two subsets of points H_{k-1}^- and H_{k-1}^+ obtained from H_{k-1} as follows: If $p = (i,j) \in H_{k-1}$, then $p' = (2i, j) \in H_{k-1}^-$ and $p'' = (2i+1, j+d_k) \in$

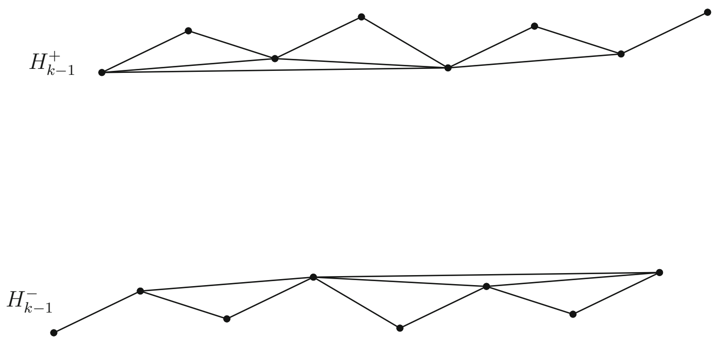


Fig. 1 H_4. The edges of H_3^+ (respectively, H_3^-) visible from *below* (respectively, *above*), are shown

H_{k-1}^{+}. The value d_k is chosen large enough such that any line ℓ passing through two points of H_{k-1}^{+} leaves all the points of H_{k-1}^{-} below it; see Fig. 1.

We say that a line segment pq joining two elements p and q of H_k is *visible from below* (respectively, *above*) if there is no point of H_k below it (respectively, above it); that is there is no element r of H_k such that the vertical line through r intersects pq above r (respectively, below r). Let $B(H_k)$ be the set of line segments of H_k visible from below. The following result, which we will use later, was proved by Bárány and Valtr in [2]; see also [3]:

Lemma 1. $|B(H_k)| = 2^{k+1} - (k+2)$.

The following result is proved in [3] by using this lemma:

Theorem 1. *For every* $n = 2^k$, $k \geq 1$, *there is a point set (namely,* H_k*) such that there is a geometric graph on* H_k *with* $\binom{n}{2} - O(n \log n)$ *edges with no empty triangles.*

In other words, it is always possible to remove $O(n \log n)$ edges from the complete graph $\mathcal{K}_{H_k}$ in such a way that the remaining graph contains no empty triangles. The main idea is that by removing from $\mathcal{K}_{H_k}$ all the edges of H_{k-1}^{+} (respectively, H_{k-1}^{-}) visible from below (respectively, above), no empty triangle remains with vertices in both H_{k-1}^{+} and H_{k-1}^{-}.

Observe now that if a geometric graph has k edge-disjoint empty triangles, then we need to take at least k edges away from G for the graph that remains to contain no empty triangles. It follows now that the complete graph $\mathcal{K}_{H_k}$ has at most $O(n \log n)$ edge-disjoint empty triangles. Thus, we have proved

Theorem 2. *There is a point set, namely,* H_k*, such that any set of edge-disjoint empty triangles of* H_k *contains at most* $O(n \log n)$ *elements.*

Clearly, for any point set P, the size of the largest set of edge-disjoint triangles of P is at least linear. We conjecture

Conjecture 1. Any point set P in general position always contains a set with at least $O(n \log n)$ edge-disjoint empty triangles.

3 Covering the Triangles of Point Sets with Triangulations

An empty triangle t of a point set P is covered by a triangulation T of P if one of the faces of T is t. In this section, we consider the following problem:

Problem 2. How many triangulations of a point set are needed such that each empty triangle of P is covered by at least one triangulation?

This problem, which is interesting in its own right, will help us in finding point sets for which $\delta(P)$ is large. We start by studying Problem 2 for point sets in convex position, and then for point sets in general position.

3.1 Points in Convex Position

All point sets P considered in this subsection will be assumed to be in convex position, and their elements labeled $\{p_0, \ldots, p_{n-1}\}$ in counterclockwise order around the boundary of $\mathrm{CH}(P)$. Since any triangulation of a point set of n points in convex position corresponds to a triangulation of a regular polygon with n vertices, solving Problem 2 for point sets in convex position is equivalent to solving it for point sets whose elements are the vertices of a regular polygon. Suppose then that P is the set of vertices of a regular polygon and that c is the center of such a polygon.

A triangle is called an *acute* triangle if all of its angles are smaller than $\frac{\pi}{2}$. We recall the following result in elementary geometry given without proof.

Observation 1. *A triangle with vertices in P is acute if and only if it contains c in its interior.*

The following result is relatively well known.

Lemma 2. *Let P be the set of vertices of a regular n-gon Q and c the center of Q. Then*

- *If n is even, c is contained in the interior of $\frac{1}{4}\left[\binom{n}{3} - \frac{n(n-2)}{2}\right]$ acute triangles of P.*
- *If n is odd, c is contained in $\left[\binom{n}{3} - \frac{n(n-1)(n-3)}{8}\right] = \frac{1}{4}\left[\binom{n}{3} + \frac{n(n-1)}{2}\right]$ acute triangles of P.*

Let $f(n) = \frac{1}{4}\left[\binom{n}{3} + \frac{n(n-2)}{2}\right]$ for n even and $f(n) = \frac{1}{4}\left[\binom{n}{3} + \frac{n(n-1)}{2}\right]$ for n odd. We now prove

Theorem 3. *$f(n)$ triangulations are always sufficient, and always necessary, to cover all the triangles of a regular polygon.*

Proof. Suppose first that n is even. For each vertex p_i of P, let $\alpha(p_i) = p_{i+\frac{n}{2}}$ be the antipodal vertex of p_i in P, where addition is taken mod n. Suppose that $\Delta(p_i, p_j, p_k)$ is an acute triangle of P (i.e., it contains c in its interior), $i < j < k$. Let $t_4(i, j, k)$ be the following set of four triangles:

$$t_4(i,j,k) = \{\Delta(p_i,p_j,p_k), \Delta(\alpha(p_i),p_j,p_k), \Delta(p_i,\alpha(p_j),p_k), \Delta(p_i,p_j,\alpha(p_k))\};$$

see Fig. 2a.

It is easy to see that all the triangles of P except those that have a right angle are in

$$\bigcup t_4(i,j,k),$$

where i, j, k range over all triples such that $\Delta(p_i, p_j, p_k)$ contains c in its interior.

On the other hand, it is easy to see that if a triangle t of P contains c in the middle of one of its edges (clearly, t is a right triangle), this edge joins two antipodal vertices of P; see Fig. 2b). Thus, we have exactly

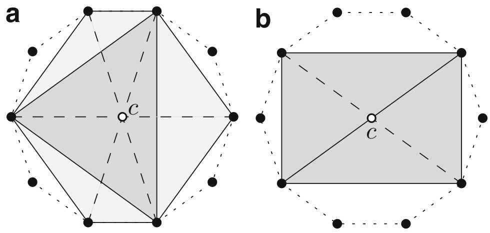


Fig. 2 (**a**) Constructing $t_4(i,j,k)$, and (**b**) pairing triangles sharing an edge, which contains c in the middle

$$\frac{n}{2} \times (n-2)$$

such triangles. It is easy to find

$$\frac{n(n-2)}{4}$$

triangulations of P such that each of them cover two of these triangles. Since each triangulation of P contains exactly one acute triangle of P or two triangles sharing an edge that contains c at its middle point, it follows that

$$\frac{1}{4}\left[\binom{n}{3} - \frac{n(n-2)}{2}\right] + \frac{n(n-2)}{4} = \frac{1}{4}\left[\binom{n}{3} + \frac{n(n-2)}{2}\right]$$

triangulations are necessary and sufficient to cover all the triangles of P. To show that this number of triangulations is needed, we point out that any two acute triangles of P cannot belong to the same triangulation (note that they intersect at c). Moreover, these triangulations are different from those containing right triangles. Our result follows.

A similar argument follows for n odd, except that some extra care has to be paid to the way in which we group the nonacute triangles of P around the acute triangles of P. □

Thus, the number of triangulations needed to cover all the triangles of P is asymptotically $\binom{n}{3}/4$. The next result follows trivially.

Corollary 1. *Let P be a set of n points in convex position and p any point in the interior of $CH(P)$. Then p belongs to the interior of at most $\frac{\binom{n}{3}}{4} + O(n^2)$ triangles of P.*

3.2 Covering the Empty Triangles on the Horton Set

We will now show that all the empty triangles in H_k can be covered with $O(n \log n)$ triangulations. The bound is tight.

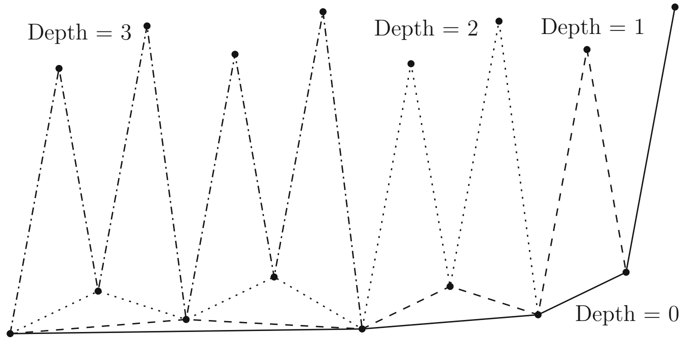


Fig. 3 The depth of an edge

Consider an edge e of H_k that is visible from below, and a vertical line ℓ that intersects e at a point q in the interior of e. The depth of e is the number of edges of H_k, visible from below, intersected by ℓ below q. It is not hard to see that the maximal depth of an edge of H_k visible from below is at most $\log n - 1$ and that this bound is tight; see Fig. 3. Moreover, it is easy to see that the union of all edges of H_k with the same depth is an x-monotone path. Now we can prove

Theorem 4. $\Theta(n \log n)$ *triangulations of* H_k *are necessary and sufficient to cover the set of empty triangles of* H_k.

Proof. Consider the sets H^+_{k-1} and H^-_{k-1}. We will show how to cover all the empty triangles of H_k with two vertices in H^+_{k-1} and one in H^-_{k-1} with $O(n \log n)$ triangulations. Label the elements of H^-_{k-1} from left to right as $p_0, \ldots, p_{\frac{n}{2}-1}$.

For each $0 \le d \le k-1$, proceed as follows: For every $p_j \in H^-_{k-1}$, join p_j to the endpoints of all the edges of H^+_{k-1} of depth d. This gives us a set $ID^+_{d,j}$ of interior-disjoint empty triangles. It is not hard to see that if $(d,j) \neq (d',j')$, then $ID^+_{d,j} \cap ID^+_{d',j'} = \emptyset$.

It is easy to see that the union of these sets covers all the empty triangles with two vertices in H^+_{k-1} and one in H^-_{k-1}. In a similar way, cover all the triangles with two vertices in H^-_{k-1}, and one in H^+_{k-1} with a family of sets $ID^-_{d,j}$.

Let ℓ_1 be the straight line connecting the leftmost point in H^+_{k-1} to the rightmost point in H^-_{k-1}, and ℓ_2 the straight line that connects the rightmost point in H^+_{k-1} with the leftmost point of H^-_{k-1}. Let q be a point slightly above the intersection point of ℓ_1 with ℓ_2.

It is clear that for each $ID^+_{d,j}$ there is exactly one empty triangle that contains q in its interior. This implies that q is contained in $\Omega(n \log n)$ empty triangles, and thus $\Omega(n \log n)$ triangulations are necessary to cover all the empty triangles in H_k.

Now we show that $O(n \log n)$ of H_k triangulations are sufficient. Consider each set $ID^+_{d,j}$ and $ID^-_{d,j}$, and complete it to a triangulation. This gives us $O(n \log n)$ triangulations that cover all the triangles with vertices in both H^+_{k-1} and H^-_{k-1}.

Take a set of triangulations $\mathcal{T}_{k-1}^{+} = \{T_1^{+}, \ldots, T_m^{+}\}$ of H_{k-1}^{+} that covers all of its empty triangles. Since H_{k-1}^{+} and H_{k-1}^{-} are isomorphic, we can find a set of triangulations $\mathcal{T}_{k-1}^{-} = \{T_1^{-}, \ldots, T_m^{-}\}$ of H_{k-1}^{-} that covers all the empty triangles of H_{k-1}^{-} such that T_i^{+} is isomorphic to T_i^{-}. For each i, we can find a triangulation T_i of H_k that contains T_i^{+} and T_i^{-} as induced subgraphs.

Thus, if $T(n)$ is the number of triangulations required to cover the empty triangles of H_k, the following recurrence holds for $n = 2^k$:

$$T(n) = T\left(\frac{n}{2}\right) + O(n \log n).$$

This solves to $T(n) = O(n \log n)$, and our result follows. □

We conclude this section with the following conjecture.

Conjecture 2. At least $\Omega(n \log n)$ triangulations are needed to cover all the empty triangles of any point set with n points.

4 A Point in Many Edge-Disjoint Triangles

The problem of finding a point contained in many triangles of a point set was solved by Boros and Füredi [4]; see also Bukh [6]. They proved

Theorem 5. *For any set P of n points in general position, there is a point in the interior of the convex hull of P contained in $\frac{2}{9}\binom{n}{3} + O(n^2)$ triangles of P. The bound is tight.*

We now study a variant to this problem, in which we are interested in finding a point in many *edge-disjoint* triangles. We consider the following.

Problem 3. Let P be a set of points in the plane in general position, and $q \notin P$ a point of the plane. What is the largest number of edge-disjoint triangles of P such that q belongs to the interior of all of them?

We start by giving some preliminary results, and then we study Problem 3 for point sets in general position and sets of vertices of regular polygons.

Given a point set P, and a point q not in P, let $\mathcal{T}(P,q)$ [or $\mathcal{T}(q)$ for short] be the set of triangles of P that contain q. We define the graph $G(P,q)$ whose vertex set is $\mathcal{T}(q)$ in which two triangles are adjacent if they share an edge; see Fig. 4. We may assume that q does not belong to any line passing through two elements of P. We now prove

Lemma 3. *The degree of every vertex of $G(P,q)$ is exactly $n-3$.*

Proof. Let $\Delta(x,y,z)$ be a triangle that contains q in its interior. Let p be any point in $P \setminus \{x,y,z\}$. Then exactly one of the triangles $\Delta(x,y,p)$, $\Delta(x,p,z)$, or $\Delta(p,y,z)$

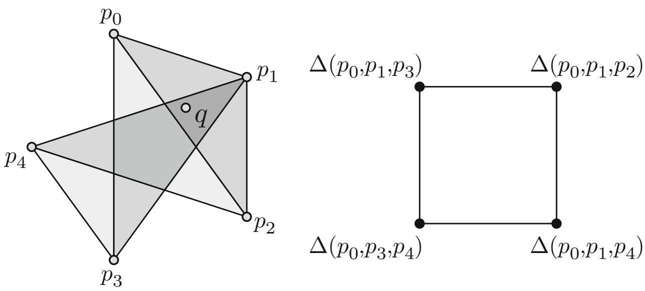
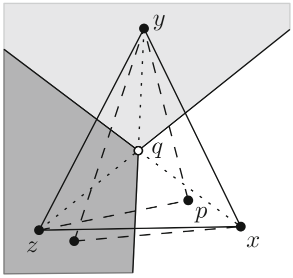


Fig. 4 $G(P,q)$

Fig. 5



contains q; see Fig. 5. That is, exactly one of $\Delta(x,y,p)$, $\Delta(x,p,z)$, or $\Delta(p,y,z)$ belongs to $\mathcal{T}(q)$. Our result follows. □

Observe now that finding sets of edge-disjoint triangles that contain q is equivalent to finding independent sets in $G(P,q)$. Let $\tau(P,q)$ (or $\tau(q)$ for short) be the largest number of edge-disjoint triangles on P containing q. We now prove

Lemma 4.

$$\frac{|\mathcal{T}(q)|}{n-2} \leq \tau(q) \leq \frac{3|\mathcal{T}(q)|}{n}.$$

Proof. It follows from Lemma 3 that the size of the largest independent set of $G(P,q)$ is at least $\frac{|\mathcal{T}(q)|}{n-2}$. To prove our upper bound, it is sufficient to observe that if a vertex of $G(P,q)$ is not in an independent set I of $G(P,q)$, then it is adjacent to at most three vertices in it, one per each of its edges. Hence, by counting the number of edges connecting a vertex in I to another in $\mathcal{T}(q)\setminus I$, we obtain

$$(n-3)|I| \leq 3|\mathcal{T}(q)\setminus I|.$$

Our result follows. □

From Theorem 5 and Lemma 4, it is easy to see that in any set of n points in general position on the plane, there is a point q such that

$$\frac{n^2}{27}+O(n) \approx \frac{\frac{2}{9}\binom{n}{3}+O(n^2)}{n-2} \leq \tau(q) \leq \frac{3\cdot\frac{2}{9}\binom{n}{3}+O(n^2)}{n} \approx \frac{n^2}{9}+O(n).$$

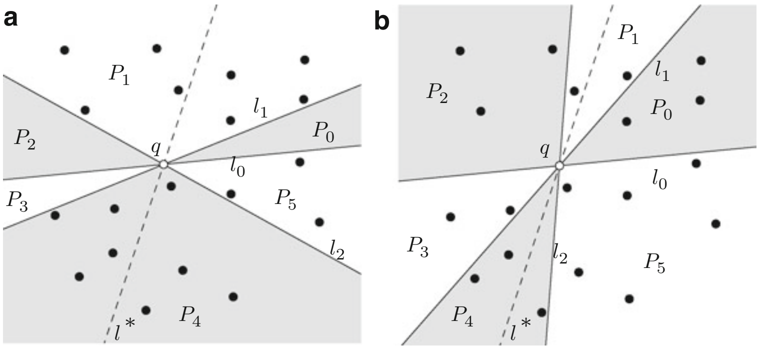


Fig. 6 Partitions of P

Thus, we have

Corollary 2. *For any point set in general position on the plane, there is a point q such that $\tau(q) \leq \frac{n^2}{9} + O(n)$.*

We now prove an even stronger result. We now prove

Proposition 1. *Let P a set of n points in general position on the plane. Then for any point $q \notin P$ of the plane, $\tau(q) \leq n^2/9$.*

Proof. Let $q \notin P$ be any point of the plane. If q is on a straight line passing through two elements of P, then by slightly moving it, q could be moved to a position in which it is contained in more edge-disjoint triangles. Thus, assume that q is not on any straight line through two elements of P.

First, we show the following lemma:

Lemma 5. *There exist three straight lines passing through q such that they partition P into six subsets $P_0, P_1, \ldots, P_5$ in counterclockwise order around q, with $|P_0| = |P_2| = |P_4|$ (we allow the possibility that $P_i = \emptyset$ for some i).*

Proof. Let l_0 be a straight line passing through q such that one of the half-planes bounded by l_0, say the one on top of it, contains an even number of elements of P. Take other straight lines l_1 and l_2 passing through q, and define the subsets P_i of P, $0 \leq i \leq 5$, as shown in Fig. 6a, where $|P_0 \cup P_1 \cup P_2|$ is even. Let l^* be a straight line passing through q, equipartitioning the elements of $P_0 \cup P_1 \cup P_2$.

Choose l_1 and l_2 such that initially $|P_0| = |P_2| = |P_3| = |P_5| = 0$. From their initial positions, rotate l_1 counterclockwise and l_2 clockwise around q in such a way that P_0 and P_2 always contain the same number of elements, and until they both reach the position of l^* at the same time, and the boundary of P_4 always contains no more than one element of P.

Initially, $|P_4| \geq 0 = |P_0|$. On the other hand, we have $|P_4| = 0 \leq |P_0|$ when l_1 and l_2 reach the position of l^*. Hence, at some point while rotating l_1 and l_2, we have that $|P_0| = |P_2| = |P_4|$; see Fig. 6b. ☐

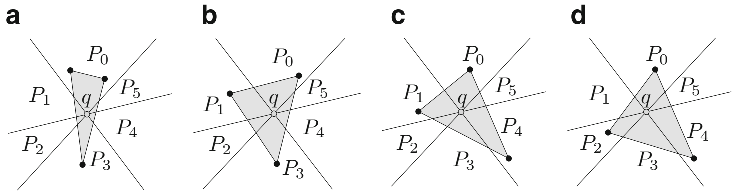


Fig. 7 Triangles in the $\mathcal{T}_{ijk}$'s

Let $P_0, P_1, \ldots, P_5$ be as in Lemma 5. Write $|P_i| = n_i$ for $0 \le i \le 5$ (we have $n_0 = n_2 = n_4$). We henceforth read indices modulo 6. Let $\mathcal{T}$ be a set of edge-disjoint triangles with vertices in P, containing q in its interior. For integers i, j, k, let $\mathcal{T}_{ijk}$ denote the set of elements of $\mathcal{T}$ such that it has one vertex in P_i, another in P_j and the other in P_k, and let $t_{ijk} = |\mathcal{T}_{ijk}|$; see Fig. 7.

Then

$$\begin{aligned}\mathcal{T} &= \left[\cup_{i=0}^{5} \mathcal{T}_{ii(i+3)}\right] \cup \left[\cup_{i=0}^{5} \mathcal{T}_{i(i+1)(i+3)}\right] \cup \left[\cup_{i=0}^{5} \mathcal{T}_{i(i+1)(i+4)}\right] \cup \left[\cup_{i=0}^{5} \mathcal{T}_{i(i+2)(i+4)}\right] \\ &= \left[\cup_{i=0}^{5} \mathcal{T}_{ii(i+3)}\right] \cup \left[\cup_{i=0}^{5} \mathcal{T}_{i(i+2)(i+5)}\right] \cup \left[\cup_{i=0}^{5} \mathcal{T}_{i(i+2)(i+3)}\right] \cup \left[\cup_{i=0}^{5} \mathcal{T}_{i(i+2)(i+4)}\right].\end{aligned}$$

For integers i, j, let E_{ij} denote the set of all segments connecting an element of P_i and another of P_j. Then for each integer i, $|E_{i(i+2)}| = n_i n_{i+2}$ and $\mathcal{T}_{i(i+2)(i+3)} \cup \mathcal{T}_{i(i+2)(i+4)} \cup \mathcal{T}_{i(i+2)(i+5)}$ is the set of elements of $\mathcal{T}$ that has a side belonging to $E_{i(i+2)}$. Hence, we have

$$f(i) \equiv t_{i(i+2)(i+3)} + t_{i(i+2)(i+4)} + t_{i(i+2)(i+5)} \le n_i n_{i+2} \tag{1}$$

for each i. Similarly, by considering the cardinality of $E_{i(i+3)}$, we obtain

$$\begin{aligned} g(i) \equiv\ & 2t_{ii(i+3)} + t_{i(i+1)(i+3)} + t_{i(i+2)(i+3)} \\ & + 2t_{i(i+3)(i+3)} + t_{i(i+3)(i+4)} + t_{i(i+3)(i+5)} \le n_i n_{i+3} \end{aligned} \tag{2}$$

for each i. By (1) and (2), we have

$$\sum_{i=0}^{5} f(i) + 2\sum_{i=0}^{2} g(i) \le \sum_{i=0}^{5} n_i n_{i+2} + 2\sum_{i=0}^{2} n_i n_{i+3}. \tag{3}$$

Since $g(i) = (t_{i(i+2)(i+3)} + t_{j(j+2)(j+3)}) + (t_{j'(j'+2)(j'+5)} + t_{j''(j''+2)(j''+5)}) + 2(t_{ii(i+3)} + t_{jj(j+3)})$, where $j = i+3$, $j' = i+1$, $j'' = j'+3$,

$$\begin{aligned}\sum_{i=0}^{5} f(i) + 2\sum_{i=0}^{2} g(i) &= \sum_{i=0}^{5} (t_{i(i+2)(i+3)} + t_{i(i+2)(i+4)} + t_{i(i+2)(i+5)}) \\ &\quad + 2\sum_{i=0}^{5} (t_{i(i+2)(i+3)} + t_{i(i+2)(i+5)}) + 4\sum_{i=0}^{5} t_{ii(i+3)} \\ &= 3|\mathcal{T}| + \sum_{i=0}^{5} t_{ii(i+3)} \ \ge\ 3|\mathcal{T}|. \end{aligned} \tag{4}$$

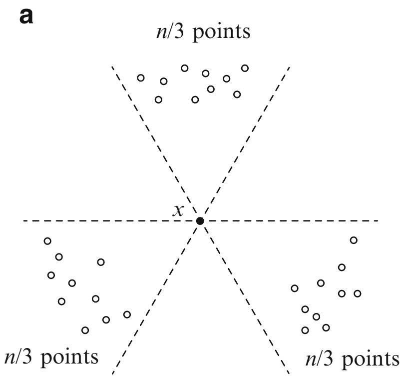


Fig. 8 A vertex set of a regular 27-gon

On the other hand, if we denote the right-hand side of (3) by S,

$$\begin{aligned} S &= (n_0n_2 + n_2n_4 + n_4n_0) + (n_1n_3 + n_3n_5 + n_5n_1) \\ &\quad +2(n_0n_3 + n_2n_5 + n_4n_1) \\ &= \frac{l^2}{3} + \frac{2lm}{3} + (n_1n_3 + n_3n_5 + n_5n_1), \end{aligned} \tag{5}$$

where $l = n_0 + n_2 + n_4$ (recall that $n_0 = n_2 = n_4$) and $m = n_1 + n_3 + n_5$. Since $n_1n_3 + n_3n_5 + n_5n_1 = [m^2 - (n_1^2 + n_3^2 + n_5^2)]/2$ and since $n_1^2 + n_3^2 + n_5^2 \geq m^2/3$ with equality if and only if $n_1 = n_3 = n_5$, we have $n_1n_3 + n_3n_5 + n_5n_1 \leq m^2/3$. From this and (5), it follows that

$$S \leq \frac{l^2}{3} + \frac{2lm}{3} + \frac{m^2}{3} = \frac{(l+m)^2}{3} = \frac{n^2}{3}. \tag{6}$$

Now combining (3), (4) and (6), we obtain $|\mathcal{T}| \leq n^2/9$, as desired. $\square$

To achieve the equality, it is necessary that $n_0 = n_2 = n_4$ and $n_1 = n_3 = n_5$ for some partition (Fig. 8).

We now prove

Proposition 2. *Let n be a positive integer and P a set of n points in general position on the plane. Then there exists a point q on the plane such that $\tau(q) \geq \frac{n^2}{12}$.*

Proof. We use the following lemma, which was proved by Ceder [7] (see also [5]) and applied by Bukh [6] to obtain a lower bound of $\max_q |\mathcal{T}(q)|$ for given P:

Lemma 6. *There exist three straight lines such that they intersect at a point q and partition the plane into six open regions each of which contains at least $n/6 - 1$ elements of P.*

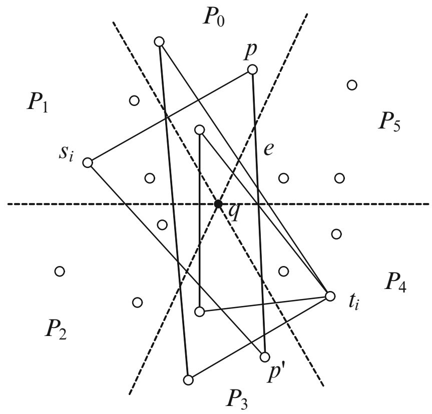


Fig. 9 Matching M_i (*bold lines*) and triangles using edges of M_i

Let q be as in Lemma 6. We may assume that q is not on any straight line passing through two elements of P. Let $m = \lceil n/6 \rceil - 1$ and denote by $D_0, D_1, \ldots, D_5$ the six regions in counterclockwise order around q. For each $0 \le i \le 5$, let P_i be a subset of $P \cap D_i$ with $|P_i| = m$; see Fig. 9.

Now consider the geometric complete bipartite graph with vertex set $P_0 \cup P_3$ and edge set $E = \{pp' \,|\, p \in P_0,\, p' \in P_3\}$. As a consequence of a well-known result in graph theory, E can be decomposed into m subsets M_i, $0 \le i \le m-1$, such that each M_i is a perfect matching, i.e., consisting of m independent edges. Let $P_1 = \{s_1, s_2, \ldots, s_m\}$ and $P_4 = \{t_1, t_2, \ldots, t_m\}$. For each i and each element $e = pp' \in M_i$, where $p \in P_0$ and $p' \in P_3$, let u_i denote either s_i or t_i according to whether $pp' \cap D_1 = \emptyset$ or $pp' \cap D_4 = \emptyset$. Then $\triangle(p, p', u_i)$ contains q in its interior. Observe that all of the m triangles in $\mathcal{T}_i = \{\triangle(p, p', u_i) \,|\, e = pp' \in M_i\}$ are edge-disjoint, and all of the m^2 triangles in $\mathcal{T}_{03} = \cup_{i=0}^{m} \mathcal{T}_i$ are edge-disjoint as well.

Define the sets $\mathcal{T}_{14}$ and $\mathcal{T}_{25}$ of triangles similarly (the elements of $\mathcal{T}_{14}$ are triangles with one vertex in P_1, another in P_4, and the other in $P_2 \cup P_5$, while the elements of $\mathcal{T}_{25}$ are triangles with one vertex in P_2, another in P_5, and the other in $P_3 \cup P_0$). It can be observed that all of the $3m^2 = n^2/12 - O(n)$ triangles in $\mathcal{T}_{03} \cup \mathcal{T}_{14} \cup \mathcal{T}_{25}$ are edge-disjoint. $\square$

Thus by using Corollary 2, Proposition 1, and Proposition 2, we have

Theorem 6. *In any point set in general position, there is a point q such that $\frac{n^2}{12} \le \tau(q) \le \frac{n^2}{9}$. Moreover, for any q, $\tau(q) \le \frac{n^2}{9}$.*

4.1 Regular Polygons

By Theorem 6, any point in the interior of the convex hull of a point set is contained in at most $n^2/9$ edge-disjoint triangles of P. It is also easy to construct point sets for which that bound is tight; see Fig. 8a). In fact, the point sets in that figure can be chosen in convex position.

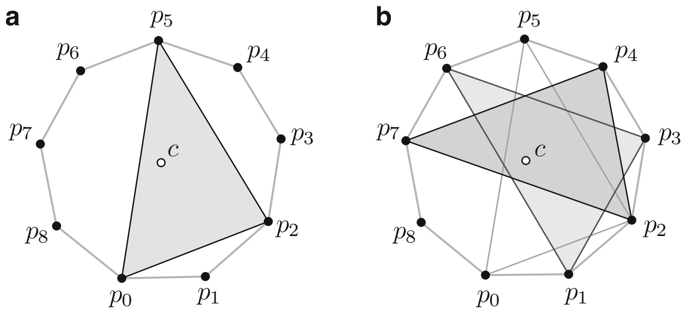


Fig. 10 (**a**) The triple $(1,2,3)$, and p_0 determine $\Delta(p_0,p_2,p_5)$. (**b**) $S(1,2,3)$ is obtained by rotating $\Delta(p_0,p_2,p_5)$, obtaining a set of 9 edge-disjoint triangles

We now show that the bound in Theorem 6 is also achieved when P is the set of vertices of a regular polygon. We found proving this result to be a challenging problem. In what follows, we will assume that $n = 9m$, $m \geq 1$.

Let (a_i,b_i,c_i) be an ordered set of integers. We call (a_i,b_i,c_i) a *triangular triple* if it satisfies the following conditions:

(a) a_i, b_i, and c_i are all different,
(b) $a_i + b_i + c_i = n-3$, and
(c) $1 \leq a_i,b_i,c_i \leq \frac{n-3}{2}$.

Observe that for any vertex p_r of P, a triangular triple (a_i,b_i,c_i), defines a triangle $\Delta(p_r,p_{r+a_i+1},p_{r+a_i+b_i+2})$ of P. Moreover, condition c) above ensures that $\Delta(p_r,p_{r+a_i+1},p_{r+a_i+b_i+2})$ is acute, and hence it contains the center c of P. Note that since $a_i+b_i+c_i = n-3$, $p_r = p_{r+a_i+b_i+c_i+3}$, addition taken mod n. Thus, the edges of $\Delta(p_r,p_{r+a_i+1},p_{r+a_i+b_i+2})$ skip a_i, b_i, and c_i vertices of P, respectively; see Fig. 10a.

Let $S(a_i,b_i,c_i) = \{\Delta(p_r,p_{r+a_i+1},p_{r+a_i+b_i+2}) : p_r \in P\}$. The set $S(a_i,b_i,c_i)$ can be seen as the set of triangles obtained by rotating $\Delta(p_0,p_{0+a_i+1},p_{0+a_i+b_i+2})$ around the center of P; see Fig. 10b. The next observation will be useful.

Observation 2. *Let (a_i,b_i,c_i) and (a_j,b_j,c_j) be triangular triples of P such that $\{a_i,b_i,c_i\} \cap \{a_j,b_j,c_j\} = \emptyset$, $i \neq j$. Then all of the triangles in $S(a_i,b_i,c_i) \cup S(a_j,b_j,c_j)$ are edge-disjoint.*

Consider a set $C = \{(a_0,b_0,c_0),\ldots,(a_{k-1},b_{k-1},c_{k-1})\}$ of ordered triangular triples. We say that C is a *generating set* of triangular triples if the following condition holds:

$$\{a_i,b_i,c_i\} \cap \{a_j,b_j,c_j\} = \emptyset,\ i \neq j.$$

$(4,8,12)$	$(7,15,20)$	$(10,22,28)$	$(13,29,36)$	$(16,36,44)$
$(5,9,10)$	$(8,13,21)$	$(11,20,29)$	$(14,27,37)$	$(17,34,45)$
$(6,7,11)$	$(9,16,17)$	$(12,18,30)$	$(15,25,38)$	$(18,32,46)$
	$(10,14,18)$	$(13,23,24)$	$(16,23,39)$	$(19,30,47)$
	$(11,12,19)$	$(14,21,25)$	$(17,30,31)$	$(20,28,48)$
		$(15,19,26)$	$(18,28,32)$	$(21,37,38)$
		$(16,17,27)$	$(19,26,33)$	$(22,35,39)$
			$(20,24,34)$	$(23,33,40)$
			$(21,22,35)$	$(24,31,41)$
				$(25,29,42)$
				$(26,27,43)$

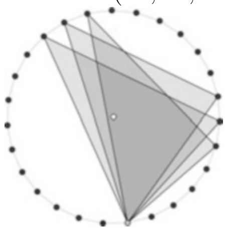


Fig. 11 Triangular triples for $n = 27, 45, 63, 81$ and 99

Note that $|S(a_i, b_i, c_i)| = n$, and thus

$$\bigcup_{(a_i,b_i,c_i)\in C} S(a_i, b_i, c_i)$$

contains nk edge-disjoint triangles containing the center P. Our task is now that of finding a generating set of as many triangular triples as possible.

Theorem 7. *Let P be the set of vertices of a regular polygon with $n = 9m$ vertices, and let c be its center. Then if m is odd, then $|\tau(c)| \geq \frac{n^2}{9}$, and if m is even, then $|\tau(c)| \geq \frac{n^2}{9} - n$.*

Proof. The proof when m is odd proceeds by constructing a generating set C with $\frac{n}{9}$ triangular triples. Let $k = \frac{9m-3}{6}$ and $k' = k + 2m - 1$. Given $i \in \{0, 1, \ldots, m-1\}$, we define the ith ordered triple (a_i, b_i, c_i) as follows (see Fig. 11):

$$\begin{aligned}
a_i &= k + i,\\
b_i &= \begin{cases} k' - 2i - 1 & \text{if} \quad i < \frac{m-1}{2},\\ k' - 2i + m - 1 & \text{if} \quad i \geq \frac{m-1}{2},\end{cases}\\
c_i &= \begin{cases} k' + i + 1 + \frac{m+1}{2} & \text{if} \quad i < \frac{m-1}{2},\\ k' + i + 1 - \frac{m-1}{2} & \text{if} \quad i \geq \frac{m-1}{2}.\end{cases}
\end{aligned}$$

We now prove that the triples (a_i, b_i, c_i) are triangular; that is, $a_i + b_i + c_i = n - 3$. Since $b_i + c_i = 2k' - i + \frac{m+1}{2}$ for all i,

$$a_i + b_i + c_i = k + 2k' + \frac{m+1}{2} = 9m - 3.$$

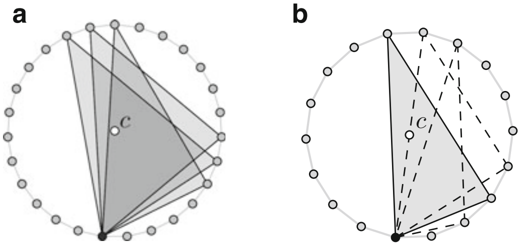


Fig. 12 (**a**) Triangular triples (a_i, b_i, c_i) for $n = 9 \cdot 3 = 27$ and (**b**) triples $(a'_i, b'_i, c'_i) = (a_i - 3, b_i - 3, c_i - 3)$ for $n = 9 \cdot 2 = 18$

It is easy to see that

$$\begin{aligned} k \leq\ & a_i \leq k+m-1, \\ k+m = k'-m+1 \leq\ & b_i \leq k', \\ k'+1 \leq\ & c_i. \end{aligned}$$

Therefore, $a_i < b_j < c_k$ for every i, j, k. Also, by construction it can be verified that $a_i \neq a_j$, $b_i \neq b_j$, and $c_i \neq c_j$ for every $i \neq j$.

$$\text{Thus, the set } \bigcup_{(a_i,b_i,c_i)\in C} \{a_i, b_i, c_i\} \text{ contains no repeated elements.}$$

Finally, note that the maximum value that can be reached by c_i occurs when $i = \frac{m-3}{2}$, and thus,

$$c_i \leq k' + 1 + \frac{m-3}{2} + \frac{m+1}{2} = k' + m = \frac{9m-3}{2}.$$

Therefore, C is a generating set of triangular triples. Thus, c is contained in at least $\frac{n^2}{9}$ edge-disjoint triangles.

The proof when m is even proceeds by also constructing a set of m triples. We use the set of triples just constructed for $m+1$ and modify it as follows: Suppose that the set of $m+1$ triples is $\{(a_0, b_0, c_0), \ldots, (a_m, b_m, c_m)\}$.

Let $a'_i = a_i - 3$, $b'_i = b_i - 3$, and $c'_i = c_i - 3$, and consider $C' = \{(a'_i, b'_i, c'_i) \mid 0 \leq i \leq m\}$. C' induces a set of triangles in P. Nevertheless, $2n$ triangles do not contain the point c in their interior; see Fig. 12. Therefore, this construction guarantees that c is contained in at least $(m+1)n - 2n = \frac{n^2}{9} - n$ edge-disjoint triangles. □

5 A Point in Many Edge-Disjoint Empty Triangles

We conclude our chapter by briefly studying the problem of the existence of a point contained in many edge-disjoint empty triangles of a point set. We point out that if we are interested only in empty triangles containing a point, it is easy to see that for any point set P, there is always a point q contained in a linear number of (not necessarily edge-disjoint) empty triangles. This follows directly from the following facts:

1. Any point set P with n elements always determines at least a quadratic number of empty triangles [2, 16].
2. We can always choose $2n-c-2$ points in the plane such that any empty triangle of P contains one of them, where c is the number of vertices of the convex hull of P; see [8, 16].

We now prove

Theorem 8. *There are point sets P such that every $q \notin P$ is contained in at most a linear number of empty edge-disjoint triangles of P.*

Proof. Let H_k, H^+_{k-1}, and H^-_{k-1} be as defined in Sect. 2. Consider any set T^+_k (respectively, T^-_k) of empty edge-disjoint triangles such that each of them has two vertices in H^+_{k-1} (respectively, H^-_{k-1}) and the other in H^-_{k-1} (respectively, H^+_{k-1}). Let $t \in T^+_k$. Then the edge of t with both endpoints in H^+_{k-1} is an edge of H^+_{k-1} visible from below. Since the triangles in T^+_k are edge-disjoint, the number of elements of T^+_k is at most the number of edges of H^+_{k-1} visible from below, which is a linear function in n. Thus, $|T^+_k| \in O(n)$. Similarly, we can prove that $|T^-_k| \in O(n)$.

Consider a point $q \in \mathrm{CH}(H_k) \setminus \mathrm{CH}(H^+_{k-1}) \cup \mathrm{CH}(H^-_{k-1})$. Clearly, any empty triangle containing q belongs to some $T^+_k \cup T^-_k$, and thus it belongs to at most a linear number of edge-disjoint triangles of H_k.

Suppose next that $q \in \mathrm{CH}(H^+_{k-1}) \cup \mathrm{CH}(H^-_{k-1})$. Suppose without loss of generality that $q \in \mathrm{CH}(H^+_{k-1})$ and that q belongs to a set S of edge-disjoint triangles of H_k. S may contain some triangles with vertices in both of H^+_{k-1} and H^-_{k-1}. There are at most a linear number of such triangles. The remaining elements in S have all of their vertices in H^+_{k-1}. Thus, the number of edge-disjoint triangles containing q satisfies

$$T(n) \le T\left(\frac{n}{2}\right) + \Theta(n),$$

and thus q belongs to at most a linear number of edge-disjoint triangles.

The first part of our result follows. To show that our bound is tight, let q be as in the proof of Theorem 4. It is easy to see that q belongs to a linear number of triangles with vertices in both H^+_k and H^-_k, and our result follows. □

We conclude with the following.

Conjecture 3. Let P be a set of n points in general position on the plane. Then there is always a point $q \notin P$ on the plane such that it is contained in at least $\log n$ edge-disjoint triangles of P.

Acknowledgements Our work was partially supported by projects MTM2009-07242, MTM2006-03909 (Spain), and SEP-CONACYT of México, Proyecto 80268.

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