A Short Proof of the Toughness of Delaunay Triangulations

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Abstract

We present a self-contained short proof of the seminal result of Dillencourt (SoCG 1987 and DCG 1990) that Delaunay triangulations, of planar point sets in general position, are 1-tough. An important implication of this result is that Delaunay triangulations have perfect matchings. Another implication of our result is a proof of the conjecture of Aichholzer et al. (2010) that at least n points are required to block any n-vertex Delaunay triangulation.

1 Introduction

Let P be a set of points in the plane that is in general position, i.e., no three points on a line and no four points on a circle. The *Delaunay triangulation* of P is an embedded planar graph with vertex set P that has a straight-line edge between two points $p, q \in P$ if and only if there exists a closed disk that has only p and q on its boundary and does not contain any other point of P. A graph is 1-*tough* if for any k, the removal of k vertices splits the graph into at most k connected components. In 1987, Dillencourt proved the toughness of Delaunay triangulations.

Theorem 1 (Dillencourt [4])**.** *Let T be the Delaunay triangulation of a set of points in the plane in general position, and let $S \subseteq V(T)$. Then $T \setminus S$ has at most $|S|$ components.*

Dillencourt's proof of Theorem 1 is nontrivial and employs a large set of combinatorial and structural properties of (Delaunay) triangulations. Using the same proof idea, he showed that if T is a Delaunay triangulation of an arbitrary point set in the plane (not necessarily in general position) then $T \setminus S$ has at most $|S|+1$ components. Combining this with Tutte's classical theorem that characterizes graphs with perfect matchings [5], implies the following well-known result.

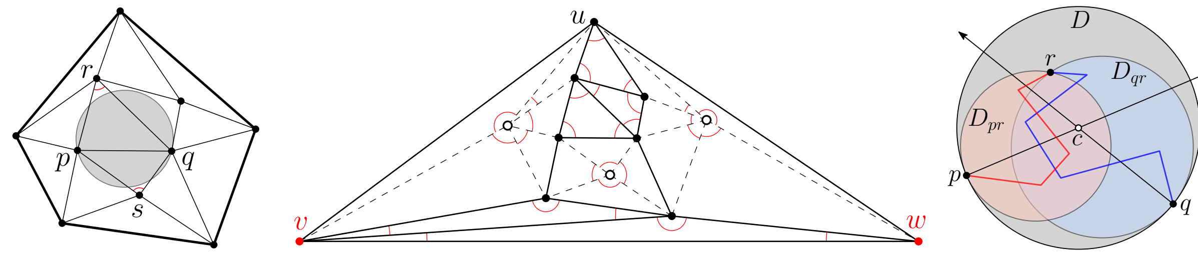
Theorem 2 (Dillencourt [4])**.** *Every Delaunay triangulation has a perfect matching.*

In this note we present a self-contained short proof of Theorem 1. To that end, we first present an upper bound on the maximum size of an independent set of T. To facilitate comparisons we use the same definitions and notations as in [4]. The number of elements of a set S is denoted by $|S|$. For a graph G, the vertex set of G is denoted by $V(G)$, and $|G| = |V(G)|$.

Every interior face of T is a triangle, and the boundary of T is a convex polygon; see Figure 1(a). An edge is called a *boundary edge* if it is on the boundary of T, and is called an *interior edge* otherwise. For any interior edge $(p, q) \in T$ between two faces pqr and pqs it holds that

$$\angle prq + \angle psq < 180. \tag{1}$$

*Part of this work has been done while the author was an NSERC postdoctoral fellow at University of Waterloo.



(a) (b) (c)

Figure 1: (a) The Delaunay triangulation T; bold segments are boundary edges. (b) The Delaunay triangulation $\mathcal{T}$; white vertices belong to I, solid edges belong to $\mathcal{T}[\mathcal{S}]$, and marked angles are distinguished angles. (c) Illustration of the proof of Theorem 4.

2 A combinatorial and a structural property

A direct implication of Theorem 1 gives the upper bound $\lfloor(|T|+1)/2\rfloor$ on the size of any independent set of T; see e.g. [1]. We present a different self-contained proof for a slightly better bound.

Theorem 3. *Let T be the Delaunay triangulation of a set of points in the plane in general position, and let I be an independent set of T. Then $|I| \leqslant \lfloor|T|/2\rfloor$, and this bound is tight.*

Proof. This upper bound is tight as any maximum independent set in the n-vertex Delaunay triangulation of Figure 2(b) has exactly $\lfloor n/2\rfloor$ vertices (regardless of the parity of n).

Now we prove the upper bound. Set $S := V(T) \setminus I$, and let u be a vertex of S that is on the boundary of T (observe that such a vertex exists). Let $v, w \notin V(T)$ be two points in the plane such that (i) T lies in the triangle (u, v, w) and (ii) neither of v and w lies in the disks that introduce edges of T; see Figure 1(b). Let $\mathcal{T}$ be the Delaunay triangulation of $V(T) \cup \{v, w\}$. Our choice of v and w ensures that any edge of T is also an edge of $\mathcal{T}$, and thus $T \subset \mathcal{T}$. Set $\mathcal{S} := S \cup \{v, w\}$. In the rest of the proof we show that $|I| \leqslant |\mathcal{S}| - 2$. This implies that $|I| \leqslant |S|$ (because $|\mathcal{S}| = |S| + 2$) which in turn implies that $|I| \leqslant \lfloor|T|/2\rfloor$ (because $|T| = |S| + |I|$, and $|I|$ and $|T|$ are integers).

To show that $|I| \leqslant |\mathcal{S}| - 2$ we use a counting argument similar to that of [4, Lemma 3.8]. Let $\mathcal{T}[\mathcal{S}]$ be the subgraph of $\mathcal{T}$ that is induced by $\mathcal{S}$. In other words, $\mathcal{T}[\mathcal{S}]$ is the resulting graph after removing vertices of I and their incident edges from $\mathcal{T}$. Since $\mathcal{T}$ is a triangulation and I does not contain boundary vertices of $\mathcal{T}$, the removal of every vertex of I creates a hole (a new face which is the union of original faces) whose boundary is a simple polygon. All edges of this polygon belong to $\mathcal{T}[\mathcal{S}]$ because I is an independent set. Therefore, $\mathcal{T}[\mathcal{S}]$ is a connected plane graph, the boundaries of its interior faces are simple polygons, and the boundary of its outer face is the triangle (u, v, w); see Figure 1(b). Each interior face of $\mathcal{T}[\mathcal{S}]$ contains either no point of I or exactly one point of I. Interior faces that do not contain any point of I are called *good faces*, and interior faces that contain a point of I are called *bad faces*. Each good face is a triangle. Let g and b denote the number of good and bad faces, respectively. Thus the number of interior faces is $g + b$.

Since $|I| = b$, it suffices to show that $b \leqslant |\mathcal{S}| - 2$. To do so, we assign to each edge $(p, q) \in \mathcal{T}[\mathcal{S}]$ certain *distinguished angles*. If (p, q) is an interior edge then we distinguish the two angles of $\mathcal{T}$ that are opposite to (p, q), and if (p, q) is a boundary edge then we distinguish the unique angle of $\mathcal{T}$ that is opposite to (p, q), as in Figure 1(b). Let d be the total measure of all distinguished angles. We compute d in two different ways: once with respect to the number of faces of $\mathcal{T}[\mathcal{S}]$ and once with respect to the number of edges of $\mathcal{T}[\mathcal{S}]$. Each good face contains three distinguished angles, their sum is 180°. The sum of the distinguished angles in each bad face is 360° because these angles are anchored at the removed vertex in the face. Therefore

$$d = 180 \cdot g + 360 \cdot b. \tag{2}$$

Now we compute d with respect to the number of edges of $\mathcal{T}[\mathcal{S}]$ which we denote by e. By Euler's formula, we have $e = |\mathcal{S}| + b + g - 1$. By Inequality (1), the sum of (at most two) distinguished angles assigned to each edge is less than 180°. Therefore

$$d < 180 \cdot e = 180 \cdot (|\mathcal{S}| + b + g - 1). \tag{3}$$

Combining (2) and (3), we have

$$180 \cdot g + 360 \cdot b < 180 \cdot (|\mathcal{S}| + b + g - 1),$$

which simplifies to $b < |\mathcal{S}| - 1$. Since b and $|\mathcal{S}|$ are integers, $b \leqslant |\mathcal{S}| - 2$. □

Our proof of Theorem 1 employs Theorem 3 and the following structural property of Delaunay triangulations presented by the author [3]. For the sake of completeness we repeat its proof.

Theorem 4. *Let T be the Delaunay triangulation of a set of points in the plane in general position. Let p and q be two vertices of T and let D be any closed disk that has on its boundary only vertices p and q. Then there exists a path, between p and q in T, that lies in D.*

Proof. The proof is by induction on the number of vertices in D. If there is no vertex of $V(T)\setminus\{p,q\}$ in the interior of D, then (p,q) is an edge of T, and so is a desired path. Assume that there exists a vertex $r \in V(T) \setminus \{p,q\}$ in the interior of D. Let c be the center of D. Consider the ray $\overrightarrow{pc}$ emanating from p and passing through c. Fix D at p and then shrink it along $\overrightarrow{pc}$ until r lies on its boundary; see Figure 1(c). Denote the resulting disk D_{pr}, and notice that it lies fully in D. Compute the disk D_{qr} in a similar fashion by shrinking D along $\overrightarrow{qc}$. The disk D_{pr} does not contain q and the disk D_{qr} does not contain p. By induction hypothesis there exists a path, between p and r in T, that lies in D_{pr}, and similarly there exists a path, between q and r in T, that lies in D_{qr}. The union of these two paths contains a path, between p and q in T, that lies in D. □

3 Proof of Theorem 1

Recall T and S. Pick an arbitrary representative vertex from each component of $T \setminus S$, and let C be the set of these vertices. The number of components is $|C|$. Consider the Delaunay triangulation T' of $S \cup C$. Observe that C is an independent set of T. We prove by contradiction that C is also an independent set of T'. Assume that there exists an edge $(c_1, c_2) \in T'$ such that $c_1, c_2 \in C$. Since T' is a Delaunay triangulation, by definition there exists a closed disk D that has only c_1 and c_2 on its boundary and does not contain any other point of $S \cup C$. Now consider T and D. By Theorem 4 there exists a path between c_1 and c_2 in T, that lies in D. Since D does not contain any point of S, all edges of this path belong to $T \setminus S$. This contradicts the fact that c_1 and c_2 belong to different components of $T \setminus S$. Therefore C is an independent set of T'. By Theorem 3, we have $|C| \leqslant |T'|/2$. This and the fact that $|T'| = |S| + |C|$ imply that $|C| \leqslant |S|$.

4 Blocking Delaunay triangulations

In this section, we use Theorem 3 and prove the conjecture of Aichholzer et al. [1] that at least n points are required to block any n-vertex Delaunay triangulation. Let P be a set of points in the plane and let T be the Delaunay triangulation of P. A point set B *blocks* or *stabs* T if in the Delaunay triangulation of $P \cup B$ there is no edge between two points of P. In other words, every disk that introduces an edge in T contains a point of B. Throughout this section we assume that $P \cup B$ is in general position.

In 2010, Aronov et al. [2] showed that $2n$ points are sufficient to block any n-vertex Delaunay triangulation, and if the vertices are in convex position then $4n/3$ points suffice. These bounds have been improved by Aichholzer et al. [1] (2010) to $3n/2$ and $5n/4$, respectively.

For the lower bound, Aronov et al. [2] showed the existence of n-vertex Delaunay triangulations that require n points to be blocked, for example see Figure 2(a) in which every disk (representing a Delaunay edge) requires a unique point to be blocked as the disks are interior disjoint. Aichholzer et al. [1] proved that at least $n-1$ points are necessary to block any n-vertex Delaunay triangulations, and stated the following conjecture.

Conjecture 1. *For any point set P in the plane in convex position, $|P|$ points are necessary and sufficient to block the Delaunay triangulation of P.*

An implication of Theorem 3 proves the necessity of $|P|$ blocking points in Conjecture 1 (even if P is in general position); the sufficiency remains open.

(a) (b)

Figure 2: (a) At least n points are required to block this n-vertex Delaunay triangulation. (b) This n-vertex Delaunay triangulation can be blocked by n points.

Theorem 5. *Let $P \cup B$ be any set of points in the plane in general position such that B blocks the Delaunay triangulation of P. Then $|B| \geqslant |P|$, and this bound is tight.*

Proof. Consider the Delaunay triangulation T of $P \cup B$. Since B blocks the Delaunay triangulation of P, the removal of B from T leaves exactly $|P|$ components each consisting of a single point of P. Thus P is an independent set of T. By Theorem 3, we have $|P| \leqslant \lfloor |T|/2 \rfloor \leqslant |T|/2$ which implies that $|B| \geqslant |P|$ (because $|T| = |P| + |B|$).

To verify the tightness of this bound, consider a set of n points in convex position where $n-1$ points are at distances approximately 1 from one point, say p, so that no four points lie on a circle. In the Delaunay triangulation of this point set, p is connected to all other points, as depicted in Figure 2(b). This Delaunay triangulation can be blocked by n points that are placed outside the convex hull: two points are placed very close to p and $n-2$ points are placed very close to the $n-2$ convex hull edges that are not incident to p. A similar placement has also been used in [1] and [2]. □

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