A Faster Algorithm for the Minimum Red-Blue-Purple Spanning Graph Problem for Points on a Circle

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Abstract

Consider a set of n points in the plane, each one of which is colored either red, blue, or purple. A red-blue-purple spanning graph (RBP spanning graph) is a graph whose vertices are the points and whose edges connect the points such that the subgraph induced by the red and purple points is connected, and the subgraph induced by the blue and purple points is connected. The minimum RBP spanning graph problem is to find an RBP spanning graph with minimum total edge length. We consider this problem for the case when the points are located on a circle. We present an algorithm that solves this problem in $O(n^2)$ time, improving upon the previous algorithm by a factor of $O(n)$.

1   Introduction

Let S be a set of n points in the plane that is partitioned into $\{R, B, P\}$. The points of R are colored red, the points of B are colored blue, and the points of P are colored purple. A *red-blue-purple spanning graph* (RBP spanning graph) on S is a graph whose vertices are the points of S and whose edges connect the points such that each of the subgraphs induced by $R \cup P$ and by $B \cup P$ are connected. In other words, if we remove the red points then the resulting subgraph is connected, and if we remove the blue points then the resulting subgraph is connected. One may think of the purple points belonging to both the red set and the blue set. The *minimum RBP spanning graph problem* is to compute an RBP spanning graph that has minimum weight (total edge length). See [3, 4] for applications of this problem. In this paper we consider the special case of this problem when the points of S are located on a circle.

In [3] it is claimed that the general case of the problem is NP-hard; this claim is based on a reduction from planar 3-SAT. However, in [4] it is claimed that the NP-hardness reduction of [3] is incorrect, and an $O(n^4)$-time exact algorithm for this problem is presented. The algorithm is based on the weighted matroid intersection. When the points of S are located on a line and given in sorted order, this problem can be solved in $O(n)$ time (see [3, 4]). If the points of S are located on a circle and given in circularly sorted order this problem can be solved in $O(n^3)$ time; specifically, it can be solved in $O(k^3 + n)$ time, where k is the number of purple points (see [3, 4]). For points on a circle, we present an algorithm that improves the running time to $O(k^2 + n)$.

2 Properties of Minimum RBP Spanning Graphs

In this section we review some properties of minimum RBP spanning graphs. Given a graph G with vertex set S, and a set $S' \subseteq S$, we denote by $G[S']$ the subgraph of G that is induced by S'.

For three sets R, B, and P of red, blue, and purple points, respectively, we denote by $G^*(R, B, P)$ a minimum RBP spanning graph on $R \cup B \cup P$; this is denoted by G^* when the triple (R, B, P) is clear from the context. As in [3, 4] we classify the edges of G^* into red, blue, and purple. An edge is *red* if it connects two red points, or a red point and a purple point. An edge is *blue* if it connects two blue points, or a blue point and a purple point. An edge is *purple* if it connects two purple points. Note that G^* does not contain any edge between a red point and a blue point. The subgraph $G^*[P]$ that is induced by the purple points is acyclic, because otherwise we could remove a purple edge from a cycle and reduce the weight of G^* without destroying the connectivity of $G^*[R \cup P]$ and $G^*[B \cup P]$. The subgraph $G^*[R \cup P]$ (resp. $G^*[B \cup P]$) is a spanning tree because otherwise we could remove a red edge (resp. blue edge) from a cycle without affecting the connectivity of $G^*[B \cup P]$ (resp. $G^*[R \cup P]$). We refer to $G^*[R \cup P]$ as the *red tree* and to $G^*[B \cup P]$ as the *blue tree*.

Every red edge in G^* is also an edge of a minimum spanning tree of $R \cup P$, because otherwise we could replace it by another red or purple edge of smaller weight. The corresponding statement holds for the blue edges. Thus, the red edges of G^* do not cross each other, and the blue edges of G^* do not cross each other. The corresponding statements do not hold for purple edges. There can be purple edges in G^* that do not occur in any minimum spanning tree of the purple points. Moreover, a purple edge in G^* can cross $\Theta(|P|)$ other purple edges [3, 4]. In [3, 4] it is shown that the maximum degree of a purple point in G^* is at most 18 and the maximum degree of a red point or a blue point is at most 6. Moreover, there exists an optimal graph in which the maximum degree of every purple point is at most 15 and the maximum degree of every red point or blue point is at most 5. The proofs for these degree constraints are inherited from the proofs of maximum degree constraints of minimum spanning trees of a point set in the plane.

3 The Algorithm

Let S be a set of n points on a circle C that are colored red, blue, or purple . Let R, B, and P denote the set of red, blue, and purple points of S, respectively. Let k denote the number of purple points, i.e., $k = |P|$. The problem is to compute a minimum RBP spanning graph for S. Although for points in the plane, and even for points in convex position, a purple edge can be crossed by other purple edges, for points on a circle, purple edges cannot be crossed by other purple edges. Based on this, Hurtado *et al.* [3, 4] presented a dynamic programming algorithm that solves this problem in $O(k^3 + n)$ time. We use a similar dynamic programming approach and improve the running time to $O(k^2 + n)$. First we review a lemma from [3, 4].

Lemma 1 (see [3, 4])**.** *Let S be a set of points on a circle, each one of which is colored either red, blue, or purple. Let G^* be a minimum RBP spanning graph for S. Then the following statements holds.*

1. *No purple edge of G^* can cross any other edge of G^*.*

2. *No red or blue edge of G^* can cross any segment between two purple points (which are not necessarily connected by an edge in G^*).*

3. *For any purple point p in S, let p' be the point on the circle diametrically opposite to p, and let SC be any of the two closed semicircles containing both p and p'. Then in G^*, p has at most one purple neighbor in SC, and thus at most two purple neighbors in total.*

3.1 The Dynamic Programming Algorithm

In this section we give an overview of the dynamic programming algorithm presented in [3, 4]. Assume that the points of S are circularly sorted. Let $p_1, \ldots, p_k$ be the purple points in clockwise order. For any $1 \leqslant i \leqslant k$, let S_i be the set of red and blue points between p_i and p_{i+1}. Assume that all indices are taken modulo k.

Let G^* be a minimum RBP spanning graph for S. By Lemma 1 no edge of G^* that is incident to a point in S_i can cross segment $p_i p_{i+1}$ ($p_i p_{i+1}$ is not necessarily an edge of G^*). Thus, a solution for each set S_i can be computed independently. Moreover, this is analogous to the case when the points are on a line, and thus, it can be solved in linear time for all sets S_i.

For any two purple points p_i and p_j, Lemma 1 guarantee that if $p_i p_j$ is an edge in G^* then it cannot be crossed by any other edge of G^*. This introduces two independent subproblems, one to the left of the oriented segment $p_i p_j$, and one to the right. Each subproblem has four different types PC, RC, BC, and NC. In the PC-type, p_i and p_j are connected in both red and blue subgraphs. In the RC-type, p_i and p_j are connected in the red subgraph but disconnected in the blue subgraph; any solution for this type must connect p_i and p_j in its blue subgraph. In the BC-type, p_i and p_j are disconnected in the red subgraph but connected in the blue subgraph. In the NC-type, p_i and p_j are neither connected in the red subgraph nor in the blue subgraph. The algorithm maintains four tables, PC, RC, BC, and NC, each of size $O(k^2)$, that are indexed by pairs of purple points. Each entry $[i, j]$ of each table, stores the length of a minimum RBP spanning graph of the corresponding type for the point set $\{p_i, p_{i+1}, \ldots, p_j\}$. Based on this, the length of an optimal solution can be found as

$$\min_{2 \leqslant j \leqslant k} \{PC[1, j] + NC[j, 1], NC[1, j] + PC[j, 1], RC[1, j] + BC[j, 1], BC[1, j] + RC[j, 1]\}.$$

Let $\Lambda \in \{P, R, B, N\}$. The entries of each table are filled in order, so that when it is time to compute the value of an entry $\Lambda[i, j]$, all the entries corresponding to smaller problems, i.e., subproblems introduced by purple pairs to the left of the oriented segment $p_i p_j$, have already been computed. In order to fill entry $\Lambda C[i, j]$, where $1 \leqslant i < j \leqslant k$, the following two cases are considered, and the one with minimum cost will be stored in $\Lambda C[i, j]$. See [3, 4] for more details.
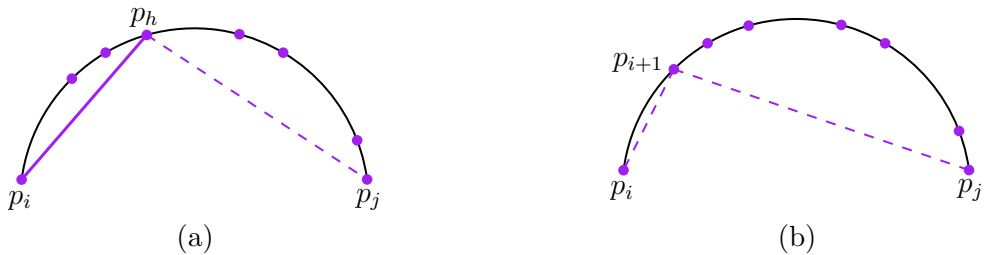

Figure 1: Solving subproblem (i, j): (a) p_i is connected to a purple point p_h, and (b) p_i is not connected to any purple point.

1. p_i is connected to some purple point(s) in an optimal solution of the subproblem (i, j); recall that by Lemma 1, p_i can be connected to at most two purple points. Let p_h be the one in the sequence $p_{i+1}, \ldots, p_j$ that is closer to p_j, see Figure 1(a). Note that p_i and p_h

are connected in both red and blue subgraphs. Therefore, p_h and p_j must be connected in the same way as p_i and p_j. Since we do not know p_h, we try all possible candidates and keep the one minimizing the cost, i.e., $\Lambda C[i, j] = \min_{i+1 \leqslant h \leqslant j}\{PC[i, h] + \Lambda C[h, j] + |p_i p_h|\}$. This case takes $O(j - i)$ time.

2. p_i is not connected to other purple points in any optimal solution of the subproblem (i, j). Now Consider p_{i+1}. By Lemma 1, in an optimal solution no edge can cross the segment $p_i p_{i+1}$. Since no purple edge is incident to p_i, the segment $p_{i+1} p_j$ cannot be crossed either; see Figure 1(b). Therefore, an optimal solution for the subproblem (i, j) can be computed by combining the solutions associated to the subproblems $(i, i+1)$ and $(i+1, j)$. See [3, 4] for more details. This case takes $O(1)$ time.

Based on the description above, the total running time of the algorithm is $O(k^3 + n)$.

3.2   Improving the Running Time

In this section we show how to improve the running time of the algorithm presented in Section 3.1 to $O(k^2 + n)$. First we prove Lemma 2 which plays an important role in this regard.

A *chord* of a circle is a straight line segment whose endpoints lie on the circle. For any two points p and q on C let $SC(p, q)$ denote the smaller arc of C that is determined by the chord pq.

Lemma 2. *Let S be a set of red, blue, and purple points located on a circle. Let P be the set of purple points. Let a and b be any two points of P such that $SC(a, b)$ contains at least two points of $P \setminus \{a, b\}$. Let $a^*, b^* \in P$ be the purple neighbors of a and b on $SC(a, b)$, respectively.*

1. *If $|aa^*| + |bb^*| < |ab|$, then ab does not belong to any minimum RBP spanning graph for S.*

2. *If $|aa^*| + |bb^*| = |ab|$, then there exists a minimum RBP spanning graph of S that does not contain ab.*

Proof. First we prove statement 1 of the lemma. The proof is by contradiction. Assume there exists a minimum RBP spanning graph G^* for S that contains ab. Recall that G^* consists of a red tree and a blue tree; moreover, the purple edges of G^* belong to both trees. For any two points p and q in S let $\delta_R(p, q)$ (resp. $\delta_B(p, q)$) denote the path between p and q in the red tree (resp. the blue tree). Consider the path $\delta_R(a^*, a)$ between a^* and a in the red tree. We differentiate between two cases depending on whether or not $\delta_R(a^*, a)$ contains the edge ab.

- $\delta_R(a^*, a)$ *does not contain the edge ab.* If $\delta_B(a^*, a)$ does not contain ab, then by replacing the purple edge ab with the purple edge a^*b we obtain an RBP spanning graph for S that is smaller than G^*; see Figure 2(a). This contradicts the minimality of G^*. Assume that $\delta_B(a^*, a)$ contains ab; see Figure 2(b).

  Claim 1: $\delta_B(a^*, a)$ *passes through b^*.*

  We prove this claim by contradiction. Let pq be the first edge of $\delta_B(a^*, a)$ that "jumps" over b^* while traversing $\delta_B(a^*, a)$ from a^* to a. Also assume p is visited before q in this traversal. See Figure 2(b). By Lemma 1, pq is neither a red nor a blue edge, because otherwise it crosses the purple segment ab^*. Thus, pq is a purple edge. By Lemma 1, pq does not cross ab. By Lemma 1, $q \neq b$, because otherwise b has at least two purple neighbors a and p. Thus, q is a purple point between b and b^* that is different from b
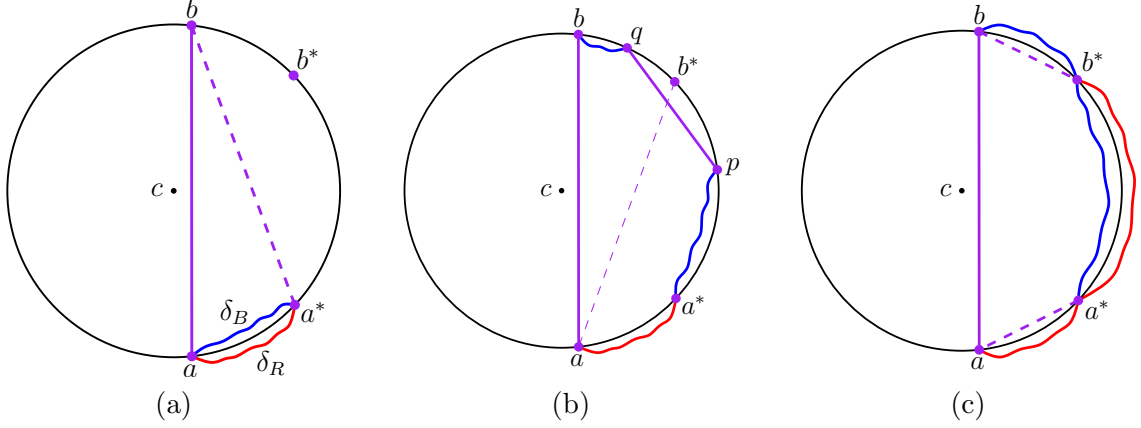
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Figure 2: Proof of Lemma 2: (a) None of $\delta_R(a^*, a)$ and $\delta_B(a^*, a)$ contains the edge ab. (b) $\delta_B(a^*, a)$ contains ab whereas $\delta_R(a^*, a)$ does not. (c) Both $\delta_B(a^*, a)$ and $\delta_R(b^*, b)$ contain ab.

and b^*. This contradicts the fact that b^* is the purple neighbor of b on $SC(a, b)$. This completes the proof of Claim 1.

Since $\delta_B(a^*, a)$ contains ab and by Claim 1 passes through b^*, $\delta_B(b^*, b)$ does not contain ab; see Figure 2(c). Now we look at $\delta_R(b^*, b)$. If $\delta_R(b^*, b)$ does not contain ab, then by replacing the purple edge ab with the purple edge ab^* we obtain an RBP spanning graph on S that is smaller than G^*. This contradicts the minimality of G^*. Assume $\delta_R(b^*, b)$ contains ab. By a similar reasoning as in the proof of Claim 1, we argue that $\delta_R(b^*, b)$ passes through a^*; see Figure 2(c). In this case, none of $\delta_R(a^*, b^*)$ and $\delta_B(a^*, b^*)$ contain ab. Thus, by replacing ab with aa^* and bb^* we obtain an RBP spanning graph for S that is smaller than G^* (Figure 2(c)). This contradicts the minimality of G^*.

- $\delta_R(a^*, a)$ *contains the edge* ab. In this case, by Claim 1, $\delta_R(a^*, a)$ passes through b^*, and hence $\delta_R(b^*, b)$ does not contain ab. The rest of the proof is similar to the proof of the previous case, with (b^*, b) playing the role of (a^*, a), and vice versa.

Now we prove statement 2 of the lemma. As we have seen in the proof of statement 1, if $|aa^*| + |bb^*| = |ab|$, in all cases we obtain an RBP spanning graph that is smaller than G^*, except for the case when we replace ab with aa^* and bb^*. Let G' be the graph that is obtained after replacing all such kind of edges. Since $|aa^*| + |bb^*| = |ab|$, G' has a weight equal to the weight of G^*. Moreover, G' does not contain ab. Thus, G' is a spanning graph that satisfies statement 2 of the lemma. $\qquad\square$

Proof. First we prove statement 1 of the lemma. The proof is by contradiction. Assume there exists a minimum RBP spanning graph G^* for S that contains ab. Recall that G^* consists of a red tree and a blue tree; moreover, the purple edges of G^* belong to both trees. Let R_a and R_b be the two red trees obtained by removing ab from G^*, such that $a \in R_a$ and $b \in R_b$. Let B_a and B_b be the two blue trees obtained in a similar way.

Claim 1: *It is not possible to have* $a^* \in R_b$ *and* $b^* \in R_a$, *or* $a^* \in B_b$ *and* $b^* \in B_a$.

We prove this claim for the case where $a^* \in R_b$ and $b^* \in R_a$; the proof for the other case is similar. By Lemma 1 (item 2) no red edge or blue edge can "jump" over a^* or b^*. Thus, in order to have $a^* \in R_b$ and $b^* \in R_a$ there must be two purple edges in G^* that cross; this contradicts Lemma 1 (item 1). This proves the claim.

By Claim 1, $a^* \in R_a$ or $b^* \in R_b$. Without loss of generality assume that $a^* \in R_a$. If $a^* \in B_a$, then by replacing the edge ab in G^* with the purple edge a^*b we obtain a valid RBP spanning graph that is smaller than G^* (note that a^*b is shorter than ab); see Figure 3(a). This contradicts the minimality of G^*. Assume that $a^* \in B_b$; see Figure 3(b). Then by Claim 1, we have $b^* \in B_b$. If $b^* \in R_b$, then by replacing the edge ab with the purple edge ab^* we obtain a valid RBP spanning graph that is smaller than G^*; see Figure 3(b). This contradicts the minimality of G^*. Assume that $b^* \in R_a$; see Figure 3(c). Then, by replacing ab with aa^* and bb^* we obtain a valid RBP spanning graph that is smaller than G^*. This contradicts the minimality of G^*.
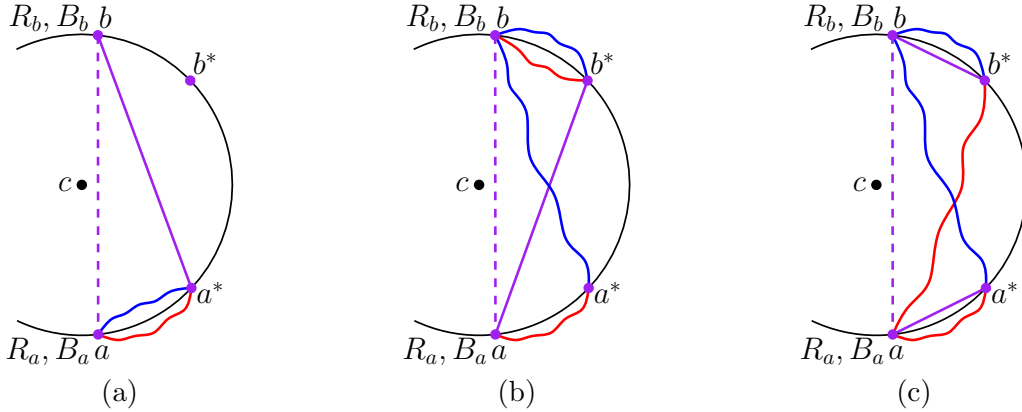

Figure 3: Proof of Lemma 2: (a) $a^* \in R_a$ and $a^* \in B_a$. (b) $a^* \in R_a$ and $a^* \in B_b$. (c) $a^* \in B_b$ and $b^* \in R_a$.

Now we prove statement 2 of the lemma. As we have seen in the proof of statement 1, if $|aa^*| + |bb^*| = |ab|$, in all cases we obtain an RBP spanning graph that is smaller than G^*, except for the case when we replace ab with aa^* and bb^*. Let G' be the graph that is obtained after replacing all such kind of edges. Since $|aa^*| + |bb^*| = |ab|$, G' has a weight equal to the weight of G^*. Moreover, G' does not contain ab. Thus, G' is a spanning graph that satisfies statement 2 of the lemma. □

We prove the following theorem in Section 4.

Theorem 1. *Let P be a set of points on a circle C. Let E_2 be the set of edges that contains an edge ab if and only if $a, b \in P$ and $|aa^*| + |bb^*| > |ab|$, where a^* and b^* are two points of P that are neighbors of a and b on the smaller arc of C that is determined by the chord ab, respectively. Then, no three edges of E_2 can pairwise cross.*

Theorem 2. *Let S be a set of n red, blue, and purple points located on a circle, and angularly sorted. A minimum RBP spanning graph of S can be computed in $O(k^2 + n)$ time, where k is the number of purple points.*

Proof. We define three sets of edges, E_0, E_1, and E_2, on the purple points as follows. Let a and b be any pair of purple points. If $SC(a, b)$ has no point of $P \setminus \{a, b\}$ then add ab to E_0. If $SC(a, b)$ contains exactly one point of $P \setminus \{a, b\}$ then add ab to E_1. Let E_2 be the set of purple edges that is defined in the statement of Theorem 1. Let $E_P = E_0 \cup E_1 \cup E_2$. As a consequence of Lemma 2 there exists an optimal solution in which all the purple edges belong to E_P. Thus, in case 1 of the dynamic programming algorithm, instead of looking at all pairs (p_i, p_h) it is enough to only consider the pairs (p_i, p_h) that are connected by an edge in E_P. Each pair (p_i, p_h) is considered only for the subproblems that have p_i or p_h as an endpoint; the

number of such subproblems is $O(k)$. Thus, the total time we spend for case 1 is $O(k|E_P|)$. Therefore the total running time of the algorithm is $O(k|E_P| + n)$.

Note that E_P can be computed in $O(k^2)$ time in the preprocessing phase. We are going to show that $|E_P| = O(k)$; this will complete the proof of the theorem. Each of E_0 and E_1 contains $k = |P|$ edges. By Theorem 1 no three edges of E_2 pairwise cross. Agarwal *et al.* [1] have shown that any graph with n vertices that can be drawn in the plane such that no three edges pairwise cross, has $O(n)$ edges. Thus, E_2 has $O(k)$ edges. Therefore, $|E_P| = O(k)$. □

4   Proof of Theorem 1

In this section we prove Theorem 1. First we prove some lemmas that will be used in the proof of the theorem.
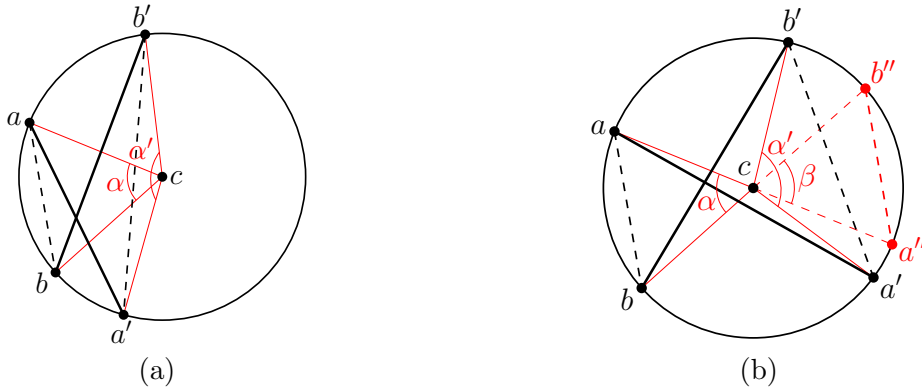
4.1   Preliminary Results

Figure 4: Proof of Lemma 3: (a) c does not lie in convex quadrilateral a, b, a', b', and (b) c lies in convex quadrilateral a, b, a', b'.

Lemma 3. *Let aa' and bb' be two intersecting chords of a circle C such that the center of C is to the left of the oriented segment aa' and to the right of the oriented segment bb'. Then, $|ab| < |a'b'|$.*

Proof. Let c be the center of C. Let $\alpha = \angle acb$ and $\alpha' = \angle a'cb'$. The triangles $\triangle abc$ and $\triangle a'b'c$ are isosceles. Based on this and assuming that the radius of C is 1, we have $|ab| = 2\sin\left(\frac{\alpha}{2}\right)$ and $|a'b'| = 2\sin\left(\frac{\alpha'}{2}\right)$. See Figure 4. Let Q be the convex quadrilateral with vertices a, b, a', and b'. If c does not lie in Q (see Figure 4(a)), then we have $\alpha < \alpha'$. This implies that $|ab| < |a'b'|$. Assume c lies in Q; see Figure 4(b). Let a'' and b'' be two points on C such that aa'' and bb'' are two diameters of C. Let $\beta = \angle a''cb''$. Note that $\beta = \alpha$. Moreover, we have $\beta < \alpha'$. This implies that that $\alpha < \alpha'$, and consequently $|ab| < |a'b'|$. □

Lemma 4. *Let b, a, and p be three points on a circle C, in this order, such that the center of C is to the right side of the oriented segment bp. Let b'' be the point such that bb'' is a diameter of C. Let SC be the smaller arc of C with endpoints p and b''. Let p' be a point on SC. Then $|ap'| - |ap| > |bp'| - |bp|$.*

Proof. Refer to Figure 5. Let $C(a, |ap|)$ be the circle of radius $|ap|$ that is centered at a, and let $C(b, |bp|)$ be the circle of radius $|bp|$ that is centered at b. Since a, p, and p' are on the same
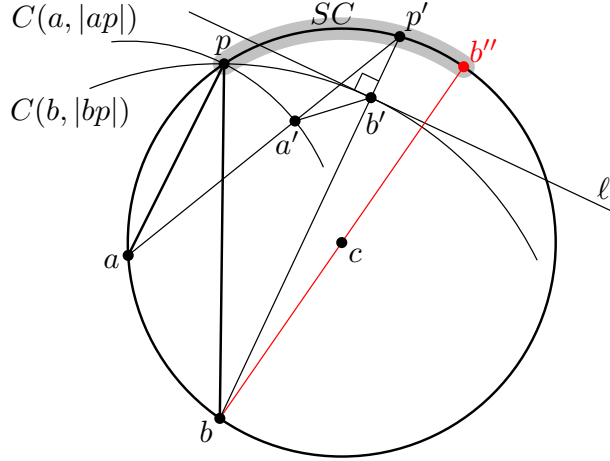
Figure 5: Proof of Lemma 4.

side of the line through bb'', we have $|ap'| > |ap|$ and $|bp'| > |bp|$. Let a' be the intersection point of ap' with $C(a, |ap|)$, and let b' be the intersection point of bp' with $C(b, |bp|)$. Then $|aa'| = |ap|$ and $|bb'| = |bp|$. Thus, $|ap'| - |ap| = |a'p'|$ and $|bp'| - |bp| = |b'p'|$. In order to prove the statement of the lemma it suffices to show that $|a'p'| > |b'p'|$. Let ℓ be the line that is tangent to $C(b, |bp|)$ at b'. Observe that a' and p' are on different sides of ℓ. Thus, in triangle $\triangle a'b'p'$, the angle $\angle a'b'p'$ is at least $\frac{\pi}{2}$. This implies that $a'p'$ is the longest side of $\triangle a'b'p'$. Therefore, $|a'p'| > |b'p'|$; this completes the proof of the lemma. $\square$

Lemma 4 can be restated as follows. If we fix the position of a and b, then by moving p towards b'', the length of ap increases more than the length of bp.
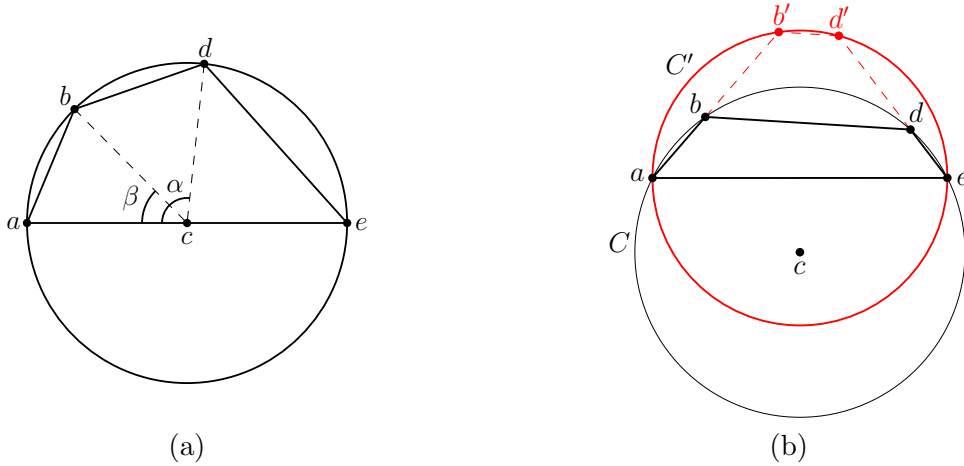

Figure 6: (a) Proof of Lemma 5 and Lemma 6. (b) Proof of Lemma 6.

Lemma 5. *Let $0 < \alpha \leqslant \pi$ be fixed, and let C be a circle that is centered at c. Let a, b, and d be three points on C, in this order, such that $\angle acd = \alpha$. Then, $|ab| + |bd|$ is maximum when $\angle acb = \angle bcd = \frac{\alpha}{2}$.*

Proof. Let $f = |ab| + |bd|$, and let $\beta = \angle acb$. See Figure 6(a). Since the triangles $\triangle acb$ and $\triangle acd$ are isosceles, we have $f = 2\sin\left(\frac{\beta}{2}\right) + 2\sin\left(\frac{\alpha-\beta}{2}\right)$. By taking the derivative of f with respect to β, we can see that f is maximum when $\beta = \frac{\alpha}{2}$, and thus, $\angle acb = \angle bcd = \frac{\alpha}{2}$. $\square$

The following is a corollary of Lemma 5.

Corollary 1. *Let C be a circle that is centered at c. Let a, b, and d be three points on C, in this order, such that $\angle acd \leqslant \frac{2\pi}{3}$. Then $|ab| + |bd| \leqslant |ac| + |dc|$.*

Proof. By Lemma 5, $|ab| + |bd|$ is maximum when $\angle acb = \angle bcd$, and thus, both these angles are at most $\frac{\pi}{3}$. This implies that $|ab| \leqslant |ac|$ and $|bd| \leqslant |dc|$, which proves the claim. $\qquad\square$

The following theorem is a restatement of Theorem 7.11 in [2].

Theorem 3 (See [2])**.** *If C_1 and C_2 are convex polygonal regions with $C_1 \subseteq C_2$, then the length of the boundary of C_1 is at most the length of the boundary of C_2.*

Lemma 6. *Let a, b, d, and e be four points on a circle C, in this order, such that the center of C is on or to the right side of the oriented segment ae. Then $|ab| + |bd| + |de| \leqslant \frac{3}{2} \cdot |ae|$.*

Proof. Without loss of generality assume C is centered at c and has radius 1. We consider two cases: (i) c is on ae, and (ii) c is not on ae. First, we prove case (i). Then, we show how to reduce case (ii) to case (i). Assume c is on ae, that is, ae is a diameter of C, and thus, $|ae| = 2$. See Figure 6(a). If we fix the position of d on C, then by Lemma 5, $|ab| + |bd|$ is maximum when $\angle acb = \angle bcd$. If we fix the position of b on C, then by Lemma 5, $|bd| + |de|$ is maximum when $\angle bcd = \angle dce$. Therefore, $|ab| + |bd| + |de|$ is maximum when $\angle acb = \angle bcd = \angle dce = \frac{\pi}{3}$, and thus, $|ab| = |bd| = |de| = |ac| = |ce|$. This implies the statement of the lemma for case (i).

Now we show how to handle case (ii). Assume c is not on ae, and thus, ae is not a diameter of C. We show how to reduce this case to case (i). Follow Figure 6(b). Let C' be the circle with diameter ae. Since C and C' intersect only at the two points a and e, we argue that b and d are in the interior of C'. Extend ab and ed to intersect C' at b' and d', respectively. Now we consider two cases depending on whether bb' and dd' intersect or not. In the former case, let c' be the intersection point of bb' and dd'. By Theorem 3 we have $|ab| + |bd| + |de| \leqslant |ac'| + |c'e|$. Since c' is in the interior of C' then $|ac'| + |c'e| \leqslant \sqrt{2} \cdot |ae|$; and we are done with this case. In the latter case, by Theorem 3 we have $|ab| + |bd| + |de| \leqslant |ab'| + |b'd'| + |d'e|$. As we have seen in case (i), $|ab'| + |b'd'| + |d'e| \leqslant \frac{3}{2} \cdot |ae|$; which completes the proof of the lemma. $\qquad\square$

4.2   Proof of Theorem

Recall that P is a set of points on a circle C. The edge set E_2 contains an edge ab if and only if $a, b \in P$ and $|aa^*| + |bb^*| > |ab|$, where a^* and b^* are the two points of P that are neighbors of a and b on the smaller arc of C that is determined by the chord ab, respectively. Without loss of generality assume C is centered at c and has radius 1. Based on the definition of E_2, the following corollary is valid.

Corollary 2. *For any edge $ab \in E_2$, we have $|aa^*| > \frac{1}{2} \cdot |ab|$ or $|bb^*| > \frac{1}{2} \cdot |ab|$.*

Lemma 7. *For any edge $ab \in E_2$, we have $|a^*b^*| < \frac{1}{2} \cdot |ab|$.*

Proof. Since $ab \in E_2$, we have $|aa^*| + |bb^*| > |ab|$. By Lemma 6 we have $|aa^*| + |a^*b^*| + |b^*b| \leqslant \frac{3}{2} \cdot |ab|$ where a, a^*, b^*, and b play the role of a, b, d, and e, respectively. This implies that $|a^*b^*| < \frac{1}{2} \cdot |ab|$. $\qquad\square$

Now we have all the tools that we need to prove Theorem 1. For the sake of contradiction assume that three edges aa', bb', and dd' of E_2 are pairwise crossing. Observe that if we remove all points of P except a, b, d, a', b', d', and then recompute E_2, the edges aa', bb', and dd' will remain in E_2. Thus, without loss of generality we assume that $P = \{a, b, d, a', b', d'\}$. Moreover,

assume a, b, d, a', b', d' appear in this order on C. Let $\triangle$ be the triangle whose vertices are the intersection points of aa', bb', and dd'. We differentiate between the following two cases: (i) c is in the interior of $\triangle$, and (ii) c is not in the interior of $\triangle$. We will get contradictions in both cases.
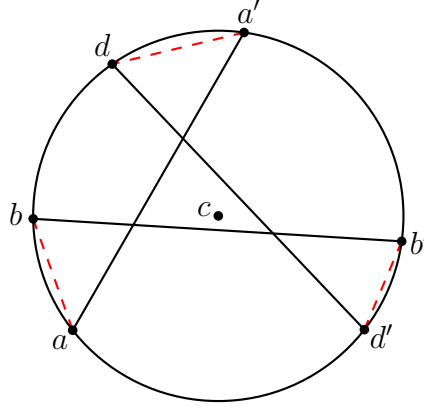

Figure 7: Proof of case (i) in Theorem 1.

First we handle case (i). Refer to Figure 7. Since aa', bb', and dd' are edges of E_2, we have $|ab| + |a'd| > |aa'|$, $|ab| + |b'd'| > |bb'|$, and $|b'd'| + |a'd| > |dd'|$. By adding up these three inequalities, we get

$$|ab| + |a'd| + |b'd'| > \frac{|aa'| + |bb'| + |dd'|}{2}. \tag{1}$$

By Lemma 3 we have $|ab| < |a'b'|$, $|a'd| < |ad'|$, and $|b'd'| < |bd|$. Adding up these three inequalities implies

$$|ab| + |a'd| + |b'd'| < |a'b'| + |ad'| + |bd|. \tag{2}$$

In view of Lemma 7, we have $|bd| < \frac{1}{2} \cdot |aa'|$, $|ad'| < \frac{1}{2} \cdot |bb'|$, and $|a'b'| < \frac{1}{2} \cdot |dd'|$. This implies

$$|bd| + |ad'| + |a'b'| < \frac{|aa'| + |bb'| + |dd'|}{2}.$$

This and Inequality (2) imply

$$|ab| + |a'd| + |b'd'| < \frac{|aa'| + |bb'| + |dd'|}{2},$$

which contradicts Inequality (1); this is a contradiction for case (i).

Now we are going to handle case (ii) where c is not in the interior of the triangle formed by the intersection of aa', bb', and dd'. Without loss of generality assume that c is on or to the right side of all the oriented segments aa', bb', and dd'; see Figure 8(a). Since aa', bb', and dd' are edges of E_2, we have $|ab| + |a'd| > |aa'|$, $|bd| + |a'b'| > |bb'|$, and $|a'd| + |b'd'| > |dd'|$. By adding up these three inequalities, we get

$$|ab| + |bd| + |a'b'| + |b'd'| + 2|a'd| > |aa'| + |bb'| + |dd'|. \tag{3}$$

Let a'' and d'' be the two points on C such that $a'a''$ and dd'' are diameters of C. By Lemma 4, $|a''b| - |ab| > |a''a'| - |aa'|$ and $|b'd''| - |b'd'| > |dd''| - |dd'|$. By adding these two inequalities to Inequality (3) we get

$$|a''b| + |bd| + |a'b'| + |b'd''| + 2|a'd| > |a''a'| + |bb'| + |dd''|. \tag{4}$$
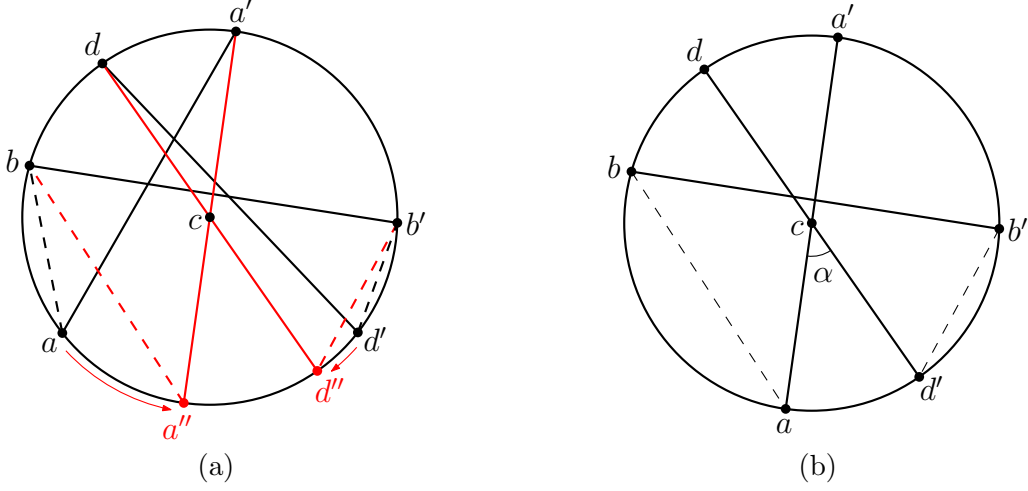
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Figure 8: Proof of case (ii) in Theorem 1.

Thus, if Inequality (3) is valid, then Inequality (4) is valid. In fact, Inequality (4) is the same as Inequality (3) where a'' and d'' play the role of a and d, respectively. Therefore, without loss of generality, from now we assume that a is on a'' and d' is on d'', that is, aa' and dd' are two diameters of C.

Let $\alpha = \angle acd' = \angle dca'$. We claim that $\alpha \leqslant \frac{\pi}{3}$. Assume $\alpha > \frac{\pi}{3}$. This implies that $\angle acd \leqslant \frac{2\pi}{3}$ and $\angle a'cd' \leqslant \frac{2\pi}{3}$. By Corollary 1 we have $|ab| + |bd| \leqslant |ac| + |cd|$ and $|a'b'| + |b'd'| \leqslant |a'c| + |cd'|$. As a consequence of Lemma 7, for edge bb' we have $2|a'd| < |bb'|$. By adding these three inequalities we get

$$|ab| + |bd| + |a'b'| + |b'd'| + 2|a'd| < |ac| + |cd| + |a'c| + |cd'| + |bb'| = |aa'| + |bb'| + |dd'|,$$

which contradicts Inequality (3). Therefore, $\alpha \leqslant \frac{\pi}{3}$.
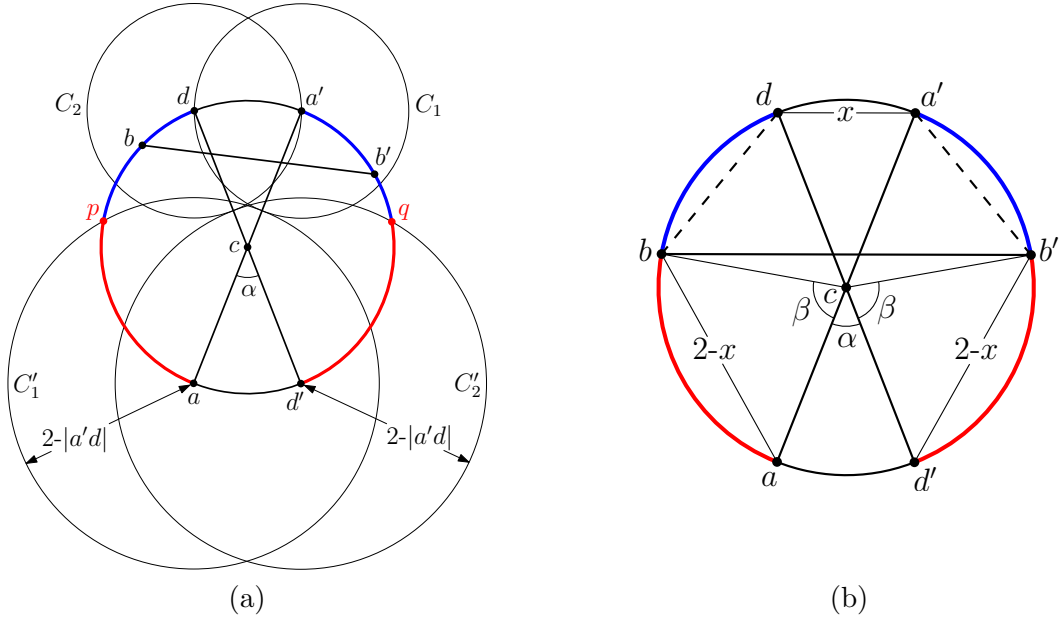

Figure 9: Proof of case (ii) in Theorem 1.

Recall that C has radius 1, and thus, $|aa'| = |dd'| = 2$. Consider two circles $C_1 = C(a', |a'd|)$ and $C_2 = C(d, |a'd|)$. Let C_1' be the circle that is centered at a and touches C_1, i.e., C_1' has

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radius $2 - |a'd|$. Similarly, let C_2' be the circle that is centered at d' and touches C_2. See Figure 9(a). Let $\widehat{ad}$ (resp. $\widehat{a'd'}$) be the smaller arc of C that has endpoints a to d (resp. a' and d'). Let p be the intersection point of C_1' and $\widehat{ad}$. Let $\widehat{ap}$ and $\widehat{pd}$ be the two sub-arcs of $\widehat{ad}$. Similarly, let q be the intersection point of C_2' and $\widehat{a'd'}$, and let $\widehat{a'q}$ and $\widehat{qd'}$ be the two sub-arcs of $\widehat{a'd'}$.

If b is in the interior of $\widehat{ap}$ then $|ab| + |a'd| < 2 = |aa'|$, which contradicts the existence of aa' in E_2. Thus $b \in \widehat{pd}$, and similarly, $b' \in \widehat{a'q}$. We are going to show that $|bd| + |a'b'| \leqslant |bb'|$; this will contradict the existence of bb' in E_2.

Since $bb' \in E_2$ we have $|bd| + |a'b'| > |bb'|$. By Lemma 4 if we move b towards p, then $|bd|$ increases more than $|bb'|$. Similarly, if we move b' towards q, then $|a'b'|$ increases more than $|bb'|$. Therefore, $|bd| + |a'b'| > |bb'|$ holds after moving b to p, and b' to q. Thus, from now we assume $b = p$ and $b' = q$. See Figure 9(b). Note that all the triangles $\triangle acb$, $\triangle bcd$, $\triangle dca'$, $\triangle a'cb'$, $\triangle b'cd'$, and $\triangle bcb'$ are isosceles. Let $x = |a'd|$, and thus, $|ab| = |b'd'| = 2 - |a'd| = 2 - x$. Note that $x = 2\sin\left(\frac{\alpha}{2}\right)$. Let $\beta = \angle acb = \angle b'cd'$. Then, $\beta = 2\arcsin\left(\frac{2-x}{2}\right) = 2\arcsin\left(1 - \sin\left(\frac{\alpha}{2}\right)\right)$. Note that $\angle bcb' = 2\pi - \alpha - 2\beta$. Thus,

$$|bb'| = 2\sin\left(\pi - \frac{\alpha}{2} - \beta\right) = 2\sin\left(\frac{\alpha}{2} + \beta\right).$$

Moreover, $\angle bcd = \angle a'cb' = \pi - \alpha - \beta$. Thus,

$$|bd| = |a'b'| = 2\sin\left(\frac{\pi - \alpha - \beta}{2}\right) = 2\cos\left(\frac{\alpha + \beta}{2}\right).$$

Now we show that $|bd| + |a'b'| \leqslant |bb'|$, which contradicts the existence of bb' in E_2. In order to show this, it suffices to prove that

$$4\cos\left(\frac{\alpha + \beta}{2}\right) \leqslant 2\sin\left(\frac{\alpha}{2} + \beta\right), \tag{5}$$

where $\beta = 2\arcsin\left(1 - \sin\left(\frac{\alpha}{2}\right)\right)$, for all $0 < \alpha \leqslant \frac{\pi}{3}$. Inequality (5) simplifies to

$$2\sqrt{1 - \sin^2\left(\frac{\alpha}{2}\right)}\sqrt{1 - \left(1 - \sin\left(\frac{\alpha}{2}\right)\right)^2} + 2\left(1 - \sin\left(\frac{\alpha}{2}\right)\right)^2 + \sin\left(\frac{\alpha}{2}\right) \leqslant 3, \tag{6}$$

where $0 < \alpha \leqslant \frac{\pi}{3}$. Let $u = \sin\left(\frac{\alpha}{2}\right)$. Then, Inequality (6) simplifies to

$$4u^2 - 4u + 1 \geqslant 0, \tag{7}$$

where $0 \leqslant u \leqslant \frac{1}{2}$; it is easy to verify that Inequality (7) is valid in this range of u. This contradicts the fact that $|bd| + |a'b'| > |bb'|$, and hence the existence of bb' in E_2; this is a contradiction for case (ii). This completes the proof of Theorem 1.

References

[1] P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir. Quasi-planar graphs have a linear number of edges. *Combinatorica*, 17(1):1–9, 1997.

[2] R. V. Benson. *Euclidean geometry and convexity*. McGraw-Hill, 1966.

[3] F. Hurtado, M. Korman, M. J. van Kreveld, M. Löffler, V. S. Adinolfi, R. I. Silveira, and B. Speckmann. Colored spanning graphs for set visualization. In *21st Int. Symp. on Graph Drawing*, pages 280–291, 2013.

[4] F. Hurtado, M. Korman, M. J. van Kreveld, M. Löffler, V. Sacristán, A. Shioura, R. I. Silveira, B. Speckmann, and T. Tokuyama. Colored spanning graphs for set visualization. *Comput. Geom., special issue in Memoriam: Ferran Hurtado*, page to appear.