Packing Plane Spanning Trees into a Point Set

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Abstract

Let P be a set of n points in the plane in general position. We show that at least $\lfloor n/3 \rfloor$ plane spanning trees can be packed into the complete geometric graph on P. This improves the previous best known lower bound $\Omega(\sqrt{n})$. Towards our proof of this lower bound we show that the center of a set of points, in the d-dimensional space in general position, is of dimension either 0 or d.

1 Introduction

In the two-dimensional space, a *geometric graph* G is a graph whose vertices are points in the plane and whose edges are straight-line segments connecting the points. A subgraph S of G is *plane* if no pair of its edges cross each other. Two subgraphs S_1 and S_2 of G are *edge-disjoint* if they do not share any edge.

Let P be a set of n points in the plane. The *complete geometric graph* $K(P)$ is the geometric graph with vertex set P that has a straight-line edge between every pair of points in P. We say that a sequence $S_1, S_2, S_3, \dots$ of subgraphs of $K(P)$ is *packed into* $K(P)$, if the subgraphs in this sequence are pairwise edge-disjoint. In a packing problem, we ask for the largest number of subgraphs of a given type that can be packed into $K(P)$. Among all subgraphs, plane spanning trees, plane Hamiltonian paths, and plane perfect matchings are of interest. Since $K(P)$ has $n(n-1)/2$ edges, at most $\lfloor n/2 \rfloor$ spanning trees, at most $\lfloor n/2 \rfloor$ Hamiltonian paths, and at most $n-1$ perfect matchings can be packed into it.

A long-standing open question is to determine whether or not it is possible to pack $\lfloor n/2 \rfloor$ plane spanning trees into $K(P)$. If P is in convex position, the answer in the affirmative follows from the result of Bernhart and Kanien [3], and a characterization of such plane spanning trees is given by Bose et al. [5]. In CCCG 2014, Aichholzer et al. [1] showed that if P is in general position (no three points on a line), then $\Omega(\sqrt{n})$ plane spanning trees can be packed into $K(P)$; this bound is obtained by a clever combination of crossing family (a set of pairwise crossing edges) [2] and double-stars (trees with only two interior nodes) [5]. Schnider [12] showed that it is not always possible to pack $\lfloor n/2 \rfloor$ plane spanning double stars into $K(P)$, and gave a necessary and sufficient condition for the existence of such a packing. As for packing other spanning structures into $K(P)$, Aichholzer et al. [1] and Biniaz et al. [4] showed a packing of 2 plane Hamiltonian cycles and a packing of $\lceil \log_2 n \rceil - 2$ plane perfect matchings, respectively.

The problem of packing spanning trees into (abstract) graphs is studied by Nash-Williams [11] and Tutte [13] who independently obtained necessary and sufficient conditions to pack k spanning trees into a graph. Kundu [10] showed that at least $\lceil (k-1)/2 \rceil$ spanning trees can be packed into any k-edge-connected graph.

In this paper we show how to pack $\lfloor n/3 \rfloor$ plane spanning trees into $K(P)$ when P is in general position. This improves the previous $\Omega(\sqrt{n})$ lower bound.

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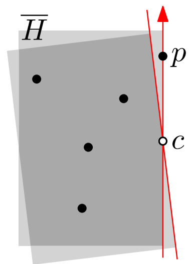
2 Packing Plane Spanning Trees

In this section we show how to pack $\lfloor n/3 \rfloor$ plane spanning tree into $K(P)$, where P is a set of $n \geqslant 3$ points in the plane in general position (no three points on a line). If $n \in \{3, 4, 5\}$ then one can easily find a plane spanning tree on P. Thus, we may assume that $n \geqslant 6$.

The *center* of P is a subset C of the plane such that any closed halfplane intersecting C contains at least $\lceil n/3 \rceil$ points of P. A *centerpoint* of P is a member of C, which does not necessarily belong to P. Thus, any halfplane that contains a centerpoint, has at least $\lceil n/3 \rceil$ points of P. It is well known that every point set in the plane has a centerpoint; see e.g. [7, Chapter 4]. We use the following corollary and lemma in our proof of the $\lfloor n/3 \rfloor$ lower bound; the corollary follows from Theorem 3 that we will prove later in Section 3.

Corollary 1. *Let P be a set of $n \geqslant 6$ points in the plane in general position, and let C be the center of P. Then, C is either 2-dimensional or 0-dimensional. If C is 0-dimensional, then it consists of one point that belongs to P, moreover n is of the form $3k+1$ for some integer $k \geqslant 2$.*

Lemma 1. *Let P be a set of n points in the plane in general position, and let c be a centerpoint of P. Then, for every point $p \in P$, each of the two closed halfplanes, that are determined by the line through c and p, contains at least $\lceil n/3 \rceil + 1$ points of P.*



Proof. For the sake of contradiction assume that a closed halfplane $\overline{H}$, that is determined by the line through c and p, contains less than $\lceil n/3 \rceil + 1$ points of P. By symmetry assume that $\overline{H}$ is to the left side of this line oriented from c to p; see the figure to the right. Since c is a centerpoint and $\overline{H}$ contains c, the definition of centerpoint implies that $\overline{H}$ contains exactly $\lceil n/3 \rceil$ points of P (including p and any other point of P that may lie on the boundary of $\overline{H}$). By slightly rotating $\overline{H}$ counterclockwise around c, while keeping c on the boundary of $\overline{H}$, we obtain a new closed halfplane that contains c but misses p. This new halfplane contains less than $\lceil n/3 \rceil$ points of P; this contradicts c being a centerpoint of P. □

Now we proceed with our proof of the lower bound. We distinguish between two cases depending on whether the center C of P is 2-dimensional or 0-dimensional. First suppose that C is 2-dimensional. Then, C contains a centerpoint, say c, that does not belong to P. Let $p_1, \dots, p_n$ be a counter-clockwise radial ordering of points in P around c. For two points p and q in the plane, we denote by $\overrightarrow{pq}$, the ray emanating from p that passes through q.

Since every integer $n \geqslant 3$ has one of the forms $3k$, $3k+1$, and $3k+2$, for some $k \geqslant 1$, we will consider three cases. In each case, we show how to construct k plane spanning directed graphs $G_1, \dots, G_k$ that are edge-disjoint. Then, for every $i \in \{1, \dots, k\}$, we obtain a plane spanning tree T_i from G_i. First assume that $n = 3k$. To build G_i, connect p_i by outgoing edges to $p_{i+1}, p_{i+2}, \dots, p_{i+k}$, then connect p_{i+k} by outgoing edges to $p_{i+k+1}, p_{i+k+2}, \dots, p_{i+2k}$, and then connect p_{i+2k} by outgoing edges to $p_{i+2k+1}, p_{i+2k+2}, \dots, p_{i+3k}$, where all the indices are modulo n, and thus $p_{i+3k} = p_i$. The graph G_i, that is obtained this way, has one cycle $(p_i, p_{i+k}, p_{i+2k}, p_i)$; see Figure 1. By Lemma 1, every closed halfplane, that is determined by the line through c and a point of P, contains at least $k+1$ points of P. Thus, all points $p_i, p_{i+1}, \dots, p_{i+k}$ lie in the closed halfplane to the left of the line through c and p_i that is oriented from c to p_i. Similarly, the points $p_{i+k}, \dots, p_{i+2k}$ lie in the closed halfplane to the left of the oriented line from c to p_{i+k}, and the points $p_{i+2k}, \dots, p_{i+3k}$ lie in the closed halfplane to the left of the oriented line from c to p_{i+2k}. Thus, all the k edges outgoing from p_i are in the convex wedge bounded by the rays $\overrightarrow{cp_i}$ and $\overrightarrow{cp_{i+k}}$, all the edges outgoing from p_{i+k} are in the convex wedge bounded by $\overrightarrow{cp_{i+k}}$ and $\overrightarrow{c_{i+2k}}$, and all the edges from p_{i+2k} are in the convex wedge bounded by $\overrightarrow{cp_{i+2k}}$ and $\overrightarrow{c_{i+3k}}$.

Therefore, the spanning directed graph G_i is plane. As depicted in Figure 1, by removing the edge (p_{i+2k}, p_i) from G_i we obtain a plane spanning (directed) tree T_i. This is the end of our construction of k plane spanning trees.

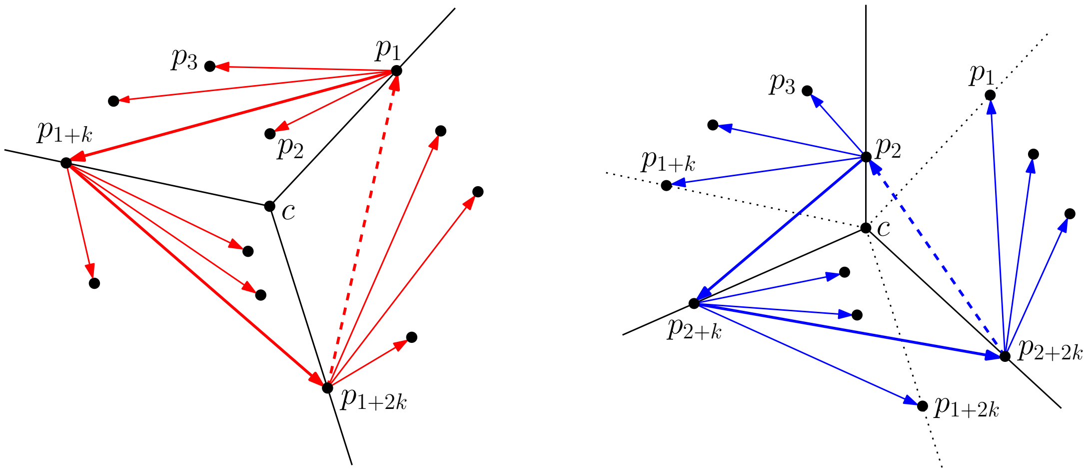


Figure 1: The plane spanning trees T_1 (the left) and T_2 (the right) are obtained by removing the edges (p_{1+2k}, p_1) and (p_{2+2k}, p_2) from G_1 and G_2, respectively.

To verify that the k spanning trees obtained above are edge-disjoint, we show that two trees T_i and T_j, with $i \neq j$, do not share any edge. Notice that the tail of every edge in T_i belongs to the set $I = \{p_i, p_{i+k}, p_{i+2k}\}$, and the tail of every edge in T_j belongs to the set $J = \{p_j, p_{j+k}, p_{j+2k}\}$, and $I \cap J = \emptyset$. For contrary, suppose that some edge (p_r, p_s) belongs to both T_i and T_j, and without loss of generality assume that in T_i this edge is oriented from p_r to p_s while in T_j it is oriented from p_s to p_r. Then $p_r \in I$ and $p_s \in J$. Since $(p_r, p_s) \in T_i$ and the largest index of the head of every outgoing edge from p_r is $r + k$, we have that $s \leqslant (r + k) \mod n$. Similarly, since $(p_s, p_r) \in T_j$ and the largest index of the head of every outgoing edge from p_s is $s + k$, we have that $r \leqslant (s + k) \mod n$. However, these two inequalities cannot hold together; this contradicts our assumption that (p_r, p_s) belongs to both trees. Thus, our claim, that $T_1, \dots, T_k$ are edge-disjoint, follows. This finishes our proof for the case where $n = 3k$.

If $n = 3k + 1$, then by Lemma 1, every closed halfplane that is determined by the line through c and a point of P contains at least $k + 2$ points of P. In this case, we construct G_i by connecting p_i to its following $k + 1$ points, i.e., $p_{i+1}, \dots, p_{i+k+1}$, and then connecting each of p_{i+k+1} and p_{i+2k+1} to their following k points. If $n = 3k + 2$, then we construct G_i by connecting each of p_i and p_{i+k+1} to their following $k + 1$ points, and then connecting p_{i+2k+2} to its following k points. This is the end of our proof for the case where C is 2-dimensional.

Now we consider the case where C is 0-dimensional. By Corollary 1, C consists of one point that belongs to P, and moreover $n = 3k + 1$ for some $k \geqslant 2$. Let $p \in P$ be the only point of C, and let $p_1, \dots, p_{n-1}$ be a counter-clockwise radial ordering of points in $P \setminus \{p\}$ around p. As in our first case (where C was 2-dimensional, c was not in P, and n was of the form $3k$) we construct k edge-disjoint plane spanning trees $T_1, \dots, T_k$ on $P \setminus \{p\}$ where p playing the role of c. Then, for every $i \in \{1, \dots, k\}$, by connecting p to p_i, we obtain a plane spanning tree for P. These plane spanning trees are edge-disjoint. This is the end of our proof. In this section we have proved the following theorem.

Theorem 1. *Every complete geometric graph, on a set of n points in the plane in general position, contains at least $\lfloor n/3 \rfloor$ edge-disjoint plane spanning trees.*

3 The Dimension of the Center of a Point Set

The *center* of a set P of $n \geqslant d+1$ points in $\mathbb{R}^d$ is a subset C of $\mathbb{R}^d$ such that any closed halfspace intersecting C contains at least $\alpha = \lceil n/(d+1) \rceil$ points of P. Based on this definition, one can characterize C as the intersection of all closed halfspaces such that their complementary open halfspaces contain less than α points of P. More precisely (see [7, Chapter 4]) C is the intersection of a finite set of closed halfspaces $\overline{H_1}, \overline{H_2}, \dots, \overline{H_m}$ such that for each $\overline{H_i}$

1. the boundary of $\overline{H_i}$ contains at least d affinely independent points of P, and
2. the complementary open halfspace H_i contains at most $\alpha - 1$ points of P, and the closure of H_i contains at least α points of P.

Being the intersection of closed halfspaces, C is a convex polyhedron. A *centerpoint* of P is a member of C, which does not necessarily belong to P. It follows, from the definition of the center, that any halfspace containing a centerpoint has at least α points of P. It is well known that every point set in the plane has a centerpoint [7, Chapter 4]. In dimensions 2 and 3, a centerpoint can be computed in $O(n)$ time [9] and in $O(n^2)$ expected time [6], respectively.

A set of points in $\mathbb{R}^d$, with $d \geqslant 2$, is said to be in *general position* if no $k+2$ of them lie in a k-dimensional flat for every $k \in \{1, \dots, d-1\}$.[1] Alternatively, for a set of points in $\mathbb{R}^d$ to be in general position, it suffices that no $d+1$ of them lie on the same hyperplane. In this section we prove that if a point set P in $\mathbb{R}^d$ is in general position, then the center of P is of dimension either 0 or d. Our proof of this claim uses the following result of Grünbaum.

Theorem 2 (Grünbaum, 1962 [8])**.** *Let $\mathcal{F}$ be a finite family of convex polyhedra in $\mathbb{R}^d$, let I be their intersection, and let s be an integer in $\{1, \dots, d\}$. If every intersection of s members of $\mathcal{F}$ is of dimension d, but I is $(d-s)$-dimensional, then there exist $s+1$ members of $\mathcal{F}$ such that their intersection is $(d-s)$-dimensional.*

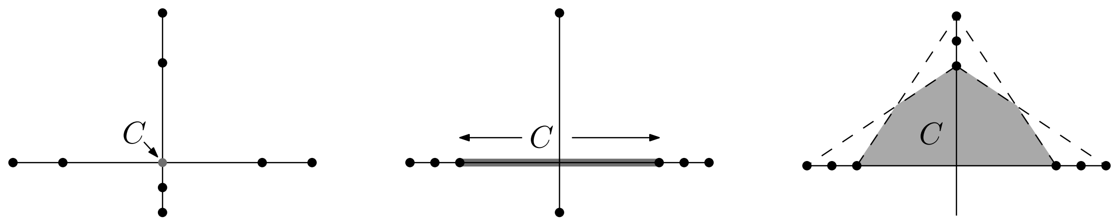


Figure 2: The dimension of a point set in the plane, that is not in general position, can be any number in $\{0, 1, 2\}$.

Before proceeding to our proof, we note that if P is not in general position, then the dimension of C can be any number in $\{0, 1, \dots, d\}$; see e.g. Figure 2 for the case where $d = 2$.

Observation 1. *For every $k \in \{1, \dots, d+1\}$ the dimension of a polyhedron defined by intersection of k closed halfspaces in $\mathbb{R}^d$ is in the range $[d-k+1, d]$.*

Theorem 3. *Let P be a set of $n \geqslant d+1$ points in $\mathbb{R}^d$, and let C be the center of P. Then, C is either d-dimensional, or contained in a $(d-s)$-dimensional polyhedron that has at least $n-(s+1)(\alpha-1)$ points of P for some $s \in \{1, \dots, d\}$ and $\alpha = \lceil n/(d+1) \rceil$. In the latter case if P is in general position and $n \geqslant d+3$, then C consists of one point that belongs to P, and n is of the form $k(d+1)+1$ for some integer $k \geqslant 2$.*

[1] A flat is a subset of d-dimensional space that is congruent to a Euclidean space of lower dimension. The flats in 2-dimensional space are points and lines, which have dimensions 0 and 1.

Proof. The center C is a convex polyhedron that is the intersection of a finite family $\mathcal{H}$ of closed halfspaces such that each of their complementary open halfspaces contains at most $\alpha - 1$ points of P [7, Chapter 4]. Since C is a convex polyhedron in $\mathbb{R}^d$, its dimension is in the range $[0, d]$. For the rest of the proof we consider the following two cases.

(a) The intersection of every $d + 1$ members of $\mathcal{H}$ is of dimension d.

(b) The intersection of some $d + 1$ members of $\mathcal{H}$ is of dimension less than d.

First assume that we are in case (a). We prove that C is d-dimensional. Our proof follows from Theorem 2 and a contrary argument. Assume that C is not d-dimensional. Then, C is $(d - s)$-dimensional for some $s \in \{1, \ldots, d\}$. Since the intersection of every s members of $\mathcal{H}$ is d-dimensional, by Theorem 2 there exist $s + 1$ members of $\mathcal{H}$ whose intersection is $(d - s)$-dimensional. This contradicts the assumption of case (a) that the intersection of every $d + 1$ members of $\mathcal{H}$ is d-dimensional. Therefore, C is d-dimensional in this case.

Now assume that we are in case (b). Let s be the largest integer in $\{1, \ldots, d\}$ such that every intersection of s members of $\mathcal{H}$ is d-dimensional; notice that such an integer exists because every single halfspace in $\mathcal{H}$ is d-dimensional. Our choice of s implies the existence of a subfamily $\mathcal{H}'$ of $s+1$ members of $\mathcal{H}$ whose intersection is d'-dimensional for some $d' < d$. Let s' be an integer such that $d' = d - s'$. By Observation 1, we have that $d' \geqslant d - s$, and equivalently $d - s' \geqslant d - s$; this implies $s' \leqslant s$. To this end we have a family $\mathcal{H}'$ with $s + 1$ members for which every intersection of s' members is d-dimensional (because $s' \leqslant s$ and $\mathcal{H}' \subseteq \mathcal{H}$), but the intersection of all members of $\mathcal{H}'$ is $(d - s')$-dimensional. Applying Theorem 2 on $\mathcal{H}'$ implies the existence of $s' + 1$ members of $\mathcal{H}'$ whose intersection is $(d - s')$-dimensional. If $s' < s$, then this implies the existence of $s' + 1 \leqslant s$ members of $\mathcal{H}' \subseteq \mathcal{H}$, whose intersection is of dimension $d - s' < d$. This contradicts the fact that the intersection of every s members of $\mathcal{H}$ is d-dimensional. Thus, $s' = s$, and consequently, $d' = d - s' = d - s$. Therefore C is contained in a $(d - s)$-dimensional polyhedron I which is the intersection of the $s + 1$ closed halfspaces of $\mathcal{H}'$. Let $H_1, \ldots, H_{s+1}$ be the complementary open halfspaces of members of $\mathcal{H}'$, and recall that each H_i contains at most $\alpha - 1$ points of P. Let $\overline{I}$ be the complement of I. Then,

$$\begin{aligned} n = |I \cup \overline{I}| &= |I \cup H_1 \cup \cdots \cup H_{s+1}| \\ &\leqslant |I| + |H_1| + \cdots + |H_{s+1}| \leqslant |I| + (s + 1)(\alpha - 1), \end{aligned}$$

where we abuse the notations I, $\overline{I}$, and H_i to refer to the subset of points of P that they contain. This inequality implies that I contains at least $n - (s + 1)(\alpha - 1)$ points of P. This finishes the proof of the theorem except for the part that P is in general position.

Now, assume that P is in general position and $n \geqslant d + 3$. By the definition of general position, the number of points of P in a $(d - s)$-dimensional flat is not more than $d - s + 1$. Since I is $(d - s)$-dimensional, this implies that

$$n - (s + 1)(\alpha - 1) \leqslant d - s + 1.$$

Notice that n is of the form $k(d + 1) + i$ for some integer $k \geqslant 1$ and some $i \in \{0, 1, \ldots, d\}$. Moreover, if i is 0 or 1, then $k \geqslant 2$ because $n \geqslant d + 3$. Now we consider two cases depending on whether or not i is 0. If $i = 0$, then $\alpha = k$. In this case, the above inequality simplifies to $k(d - s) \leqslant d - 2s$, which is not possible because $k \geqslant 2$ and $d \geqslant s \geqslant 1$. If $i \in \{1, \ldots, d\}$, then $\alpha = k + 1$. In this case, the above inequality simplifies to $(k - 1)(d - s) + i \leqslant 1$, which is not possible unless $d = s$ and $i = 1$. Thus, for the above inequality to hold we should have $d = s$ and $i = 1$. These two assertions imply that $n = k(d + 1) + 1$, and that I is 0-dimensional and consists of one point of P. Since $C \subseteq I$ and C is not empty, C also consists of one point of P. □

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