Higher-Order Triangular-Distance Delaunay Graphs: Graph-Theoretical Properties*

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Abstract. We consider an extension of the triangular-distance Delaunay graphs (TD-Delaunay) on a set P of points in general position in the plane. In TD-Delaunay, the convex distance is defined by a fixed-oriented equilateral triangle $\triangledown$, and there is an edge between two points in P if and only if there is an empty homothet of $\triangledown$ having the two points on its boundary. We consider higher-order triangular-distance Delaunay graphs, namely k-TD, which contains an edge between two points if the interior of the smallest homothet of $\triangledown$ having the two points on its boundary contains at most k points of P. We consider the connectivity, Hamiltonicity and perfect-matching admissibility of k-TD. Finally we consider the problem of blocking the edges of k-TD.

1 Introduction

The *triangular-distance Delaunay graph* of a point set P in the plane, TD-Delaunay for short, was introduced by Chew [12]. A TD-Delaunay is a graph whose convex distance function is defined by a fixed-oriented equilateral triangle. Let $\triangledown$ be a downward equilateral triangle whose barycenter is the origin and one of its vertices is on the negative y-axis. A *homothet* of $\triangledown$ is obtained by scaling $\triangledown$ with respect to the origin by some factor $\mu \geq 0$, followed by a translation to a point b in the plane: $b + \mu\triangledown = \{b + \mu a : a \in \triangledown\}$. In the TD-Delaunay graph of P, there is a straight-line edge between two points p and q if and only if there exists a homothet of $\triangledown$ having p and q on its boundary and whose interior does not contain any point of P. In other words, (p, q) is an edge of TD-Delaunay graph if and only if there exists an empty downward equilateral triangle having p and q on its boundary. In this case, we say that the edge (p, q) has the *empty triangle property.*

We say that P is in general position if the line passing through any two points from P does not make angles $0°$, $60°$, or $120°$ with horizontal. In this paper we consider point sets in general position and our results assume this pre-condition. If P is in general position, the TD-Delaunay graph of P is planar, see [7]. We define $t(p, q)$ as the smallest homothet of $\triangledown$ having p and q on its boundary. See Figure 1(a). Note that $t(p, q)$ has one of p and q at a vertex, and the other one on the opposite side. Thus,

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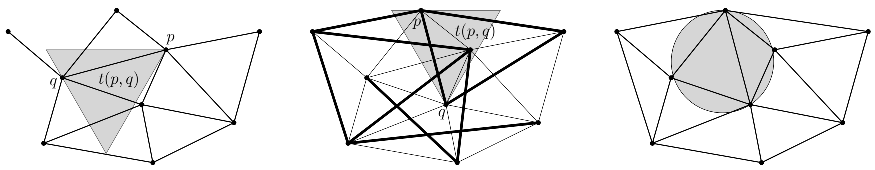
Observation 1 *Each side of $t(p,q)$ contains either p or q.*

Since every homothet of $\bigtriangledown$ with p and q on its boundary contains $t(p,q)$, the TD-Delaunay graph has an edge (p,q) iff the interior of $t(p,q)$ does not contain any point of P.



(a) (b) (c)

Fig. 1. (a) Triangular-distance Delaunay graph (0-TD), (b) 1-TD graph, the light edges belong to 0-TD as well, and (c) Delaunay triangulation.

In [4], the authors proved a tight lower bound of $\lceil \frac{n-1}{3} \rceil$ on the size of a maximum matching in a TD-Delaunay graph. In this paper we study higher-order TD-Delaunay graphs. An *order-k TD-Delaunay graph* of a point set P, denoted by k-TD, is a geometric graph which has an edge (p,q) iff the interior of $t(p,q)$ contains at most k points of P; see Figure 1(b). The standard TD-Delaunay graph corresponds to 0-TD. We consider graph-theoretic properties of higher-order TD-Delaunay graphs (connectivity, Hamiltonicity, and perfect-matching admissibility). We also consider the problem of blocking TD-Delaunay graphs.

1.1 Previous Work

A *Delaunay triangulation* (DT) of P (which does not have any four co-circular points) is a graph whose distance function is defined by a fixed circle $\bigcirc$ centered at the origin. DT has an edge between two points p and q iff there exists a homothet of $\bigcirc$ having p and q on its boundary and whose interior does not contain any point of P; see Figure 1(c). In this case the edge (p,q) is said to have the *empty circle property*. An *order-k Delaunay Graph* on P, denoted by k-DG, is defined to have an edge (p,q) iff there exists a homothet of $\bigcirc$ having p and q on its boundary and whose interior contains at most k points of P. The standard Delaunay triangulation corresponds to 0-DG.

For each pair of points $p,q \in P$ let $D[p,q]$ be the closed disk having pq as diameter, and let $L(p,q)$ be the intersection of the two open disks with radius $|pq|$ centered at p and q, where $|pq|$ is the Euclidean distance between p and q. A *Gabriel Graph* on P is a geometric graph which has an edge between two points p and q iff $D[p,q]$ does not contain any point of $P \setminus \{p,q\}$. An *order-k Gabriel Graph* on P, denoted by k-GG, is defined to have an edge (p,q) iff $D[p,q]$ contains at most k points of $P \setminus \{p,q\}$. A *Relative Neighborhood Graph* on P is a geometric graph which has an edge between two points p and q iff $L(p,q)$ does not contain any point of P. An *order-k Relative Neighborhood Graph* on P, denoted by k-RNG, is defined to have an edge (p, q) iff $L(p, q)$ contains at most k points of P. It is obvious that for a fixed point set, k-RNG is a subgraph of k-GG, and k-GG is a subgraph of k-DG.

Let $K_n(P)$ be a complete edge-weighted geometric graph on a point set P which contains a straight-line edge between any pair of points in P. For an edge (p, q) in $K_n(P)$ let $w(p, q)$ denote the weight of (p, q). A *bottleneck matching* (resp. *bottleneck Hamiltonian cycle*) in P is defined to be a perfect matching (resp. Hamiltonian cycle) in $K_n(P)$ in which the weight of the maximum-weight edge is minimized. A graph is *biconnected* if there is a simple cycle between any pair of its vertices. A *bottleneck biconnected spanning graph* of P is a spanning subgraph, $G(P)$, of $K_n(P)$ which is biconnected and in which the weight of the longest edge is minimized. For $H \subseteq G$ we denote the bottleneck of H, i.e., the length of the maximum-weight edge in H, by $\lambda(H)$.

The problem of determining whether an order-k geometric graph always has a (bottleneck) perfect matching or a (bottleneck) Hamiltonian cycle is quite of interest. If for each edge (p, q) in $K_n(P)$, $w(p, q)$ is equal the Euclidean distance between p and q, then Chang et al. [10, 11, 9] proved that a bottleneck biconnected spanning graph, a bottleneck perfect matching, and a bottleneck Hamiltonian cycle of P are contained in 1-RNG, 16-RNG, 19-RNG, respectively. This implies that 16-RNG has a perfect matching and 19-RNG is Hamiltonian. Since k-RNG is a subgraph of k-GG, the same results hold for 16-GG and 19-GG. It is known that k-GG is $(k + 1)$-connected [8] and 15-GG (and hence 15-DG) is Hamiltonian [1]. Recently, Kaiser et al. [15] proved that 10-GG is Hamiltonian. Dillencourt showed that any Delaunay triangulation (0-DG) admits a perfect matching [14] but it can fail to be Hamiltonian [13].

Given a geometric graph $G(P)$ on a set P of n points, we say that a set K of points *blocks* $G(P)$ if in $G(P \cup K)$ there is no edge connecting two points in P. Actually P is an independent set in $G(P \cup K)$. Aichholzer et al. [2] considered the problem of blocking the Delaunay triangulation (i.e. 0-DG) for P in general position. They show that $\frac{3n}{2}$ points are sufficient to block 0-DG and at least $n - 1$ points are necessary. To block 0-GG, $n - 1$ points are sufficient [3].

1.2 Our Results

In this paper we consider the bottleneck problems in P with respect to the triangular-distance. We assume that the weight of each edge (p, q) in $K_n(P)$ is equal to the area of $t(p, q)$. We define some geometric notions in Section 2. In Section 3 we prove that every k-TD graph is $(k + 1)$-connected. In addition we show that a bottleneck biconnected spanning graph of P is contained in 1-TD. Using a similar approach as in [1, 9], in Section 4 we show that a bottleneck Hamiltonian cycle of P is contained in 8-TD. In Section 5 we prove that a bottleneck perfect matching of P is contained in 6-TD. In addition we prove that 2-TD has a matching of size $\lceil \frac{(n-1)}{2} \rceil$ and 1-TD has a matching of size at least $\lceil \frac{2(n-1)}{5} \rceil$. For some configurations of P, 5-TD fails to have any bottleneck Hamiltonian cycle or bottleneck perfect matching. In Section 6 we consider the problem of blocking k-TD. We show that at least $\lceil \frac{n-1}{2} \rceil$ points are necessary and $n-1$ points are sufficient to block a 0-TD. Due to the space limitations, details of some proofs are omitted from this version of the paper.

2 Preliminaries

Bonichon et al. [6] showed that the half-Θ_6 graph of a point set P in the plane is equal to the TD-Delaunay graph of P. A half-Θ_6 graph on a point set P can be constructed in the following way. For each point p in P, let l_p be the horizontal line through p. Define l_p^γ as the line obtained by rotating l_p by γ-degrees in counter-clockwise direction around p. Actually $l_p^0 = l_p$. Consider three lines l_p^0, l_p^{60}, and l_p^{120} which partition the plane into six disjoint cones with apex p. Let $C_p^1, \dots, C_p^6$ be the cones in counter-clockwise order around p as shown in Figure 2. C_p^1, C_p^3, C_p^5 will be referred to as *odd cones*, and C_p^2, C_p^4, C_p^6 will be referred to as *even cones*. For each even cone C_p^i, connect p to the "nearest" point q in C_p^i. The *distance* between p and q, $d(p,q)$, is defined as the Euclidean distance between p and the orthogonal projection of q onto the bisector of C_p^i. See Figure 2. The resulting graph is the half-Θ_6 graph which is defined by even cones [6]. Moreover, the resulting graph is the TD-Delaunay graph defined with respect to homothets of $\bigtriangledown$. By considering the odd cones, another half-Θ_6 graph is obtained. The well-known Θ_6 graph is the union of half-Θ_6 graphs defined by odd and even cones. To construct k-TD, for each point $p \in P$ we connect p to its $(k+1)$ nearest neighbors in each even cone around p.

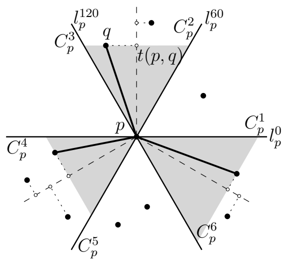


Fig. 2. Construction of the TD-Delaunay graph.

Recall that $t(p,q)$ is the smallest homothet of $\bigtriangledown$ having p and q on its boundary, i.e., $t(p,q)$ is the smallest downward equilateral triangle through p and q. Similarly we define $t'(p,q)$ as the smallest upward equilateral triangle through p and q. Clearly, the even cones correspond to downward triangles and odd cones correspond to upward triangles. We define an order on the equilateral triangles: for each two equilateral triangles t_1 and t_2 we say that $t_1 \prec t_2$ if the area of t_1 is less than the area of t_2. Since the area of $t(p,q)$ is directly related to $d(p,q)$,

$$d(p,q) < d(r,s) \quad \text{if and only if} \quad t(p,q) \prec t(r,s).$$

Observation 2 *If $t(p,q)$ contains a point r, then $t(p,r)$ and $t(q,r)$ are contained in $t(p,q)$ (see Figure 3).*

As a direct consequence of Observation 2, if a point r is contained in $t(p,q)$, then $\max\{t(p,r), t(q,r)\} \prec t(p,q)$. It is obvious that,

Observation 3 *For each two points $p, q \in P$, the area of $t(p,q)$ is equal to the area of $t'(p,q)$.*

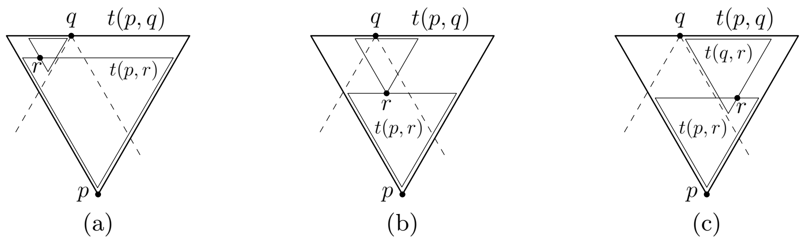


Fig. 3. Illustration of Observation 2: the triangles $t(p, r)$ and $t(q, r)$ are inside $t(p, q)$.

Thus, we define $X(p, q)$ as a regular hexagon centred at p which has q on its boundary, and its sides are parallel to l_p^0, l_p^{60}, and l_p^{120}.

Observation 4 *If $X(p, q)$ contains a point r, then $t(p, r) \prec t(p, q)$.*

For a graph $G = (V, E)$ and $K \subseteq V$, let $G - K$ be the subgraph obtained from G by removing the vertices in K, and let $o(G - K)$ be the number of odd components in $G - K$. Tutte [16] gave a characterization of the graphs which have a perfect matching. Berge [5] extended Tutte's result to a formula (known as Tutte-Berge formula) for the maximum size of a matching in a graph. In a graph G, the *deficiency*, $\text{def}_G(K)$, is $o(G - K) - |K|$. Let $\text{def}(G) = \max_{K \subseteq V} \text{def}_G(K)$.

Theorem 1 (Tutte-Berge formula; Berge [5]). *The size of a maximum matching in G is $(n - \text{def}(G))/2$.*

For an edge-weighted graph G we define the *weight sequence* of G, $\text{WS}(G)$, as the sequence containing the weights of the edges of G in non-increasing order. For two graphs G_1 and G_2 we say that $\text{WS}(G_1) \prec \text{WS}(G_2)$ if $\text{WS}(G_1)$ is lexicographically smaller than $\text{WS}(G_2)$. A graph G_1 is said to be less than a graph G_2 if $\text{WS}(G_1) \prec \text{WS}(G_2)$.

3 Connectivity

For a set P of points in general position in the plane, the TD-Delaunay graph, i.e., 0-TD, is not necessarily a triangulation [12], but it is connected and internally triangulated [4]. As shown in Figure 1(a), 0-TD may not be biconnected.

Theorem 2. *For every point set P in general position, k-TD is $(k+1)$-connected. In addition, for every k, there exists a point set P such that k-TD is not $(k+2)$-connected.*

By Theorem 2, 0-TD may not be biconnected, but 1-TD is biconnected. We show that a bottleneck biconnected spanning graph of P is contained in 1-TD.

Theorem 3. *For every point set P in general position, 1-TD contains a bottleneck biconnected spanning graph of P.*

Proof. Let $\mathcal{G}$ be the set of all biconnected spanning graphs with vertex set P. We define a total order on the elements of $\mathcal{G}$ by their weight sequence. If two elements have the same weight sequence, we break the ties arbitrarily to get a total order.

Let $G^* = (P, E)$ be a graph in $\mathcal{G}$ with minimal weight sequence. Clearly, G^* is a bottleneck biconnected spanning graph of P. We will show that all edges of G^* are in 1-TD. By contradiction suppose that some edges in E do not belong to 1-TD, and let $e = (a, b)$ be the longest one (by the area of the triangle $t(a, b)$). If the graph $G^* - \{e\}$ is biconnected, then by removing e, we obtain a biconnected spanning graph G with $\mathrm{WS}(G) \prec \mathrm{WS}(G^*)$; this contradicts the minimality of G^*. Thus, there is a pair $\{p, q\}$ of points such that any cycle between p and q in G^* goes through e. Since $(a, b) \notin$ 1-TD, $t(a, b)$ contains at least two points of P, say x and y. Let G be the graph obtained from G^* by removing the edge (a, b) and adding the edges (a, x), (b, x), (a, y), (b, y). We show that in G there is a cycle C between p and q which does not go through e. Consider a cycle C^* in G^* between two points p and q (which goes through e). If none of x and y belong to C^*, then $C = (C^* - \{(a, b)\}) \cup \{(a, x), (b, x)\}$ is a cycle in G between p and q. If one of x or y, say x, belongs to C^*, then $C = (C^* - \{(a, b)\}) \cup \{(a, y), (b, y)\}$ is a cycle in G between p and q. If both x and y belong to C^*, w.l.o.g. assume that x is between b and y in the path $C^* - \{(a, b)\}$. Consider the partition of C^* into four parts: (a) edge (a, b), (b) path δ_{bx} between b and x, (c) path δ_{xy} between x and y, and (d) path δ_{ya} between y and a. There are four cases:

1. None of p and q are on δ_{xy}. Let $C = \delta_{bx} \cup \delta_{ya} \cup \{(a, x), (b, y)\}$.
2. Both p and q are on δ_{xy}. Let $C = \delta_{xy} \cup \{(a, x), (a, y)\}$.
3. One of p, q is on δ_{xy} while the other is on δ_{bx}. Let $C = \delta_{bx} \cup \delta_{xy} \cup \{(b, y)\}$.
4. One of p, q is on δ_{xy} while the other is on δ_{ya}. Let $C = \delta_{xy} \cup \delta_{ya} \cup \{(a, x)\}$.

In all cases, C is a cycle in G between p and q. Thus, between any pair of points in G there exists a cycle, and hence G is biconnected. Since x and y are inside $t(a, b)$, by Observation 2, $\max\{t(a, x), t(a, y), t(b, x), t(b, y)\} \prec t(a, b)$. Therefore, $\mathrm{WS}(G) \prec \mathrm{WS}(G^*)$; contradicting the minimality of G^*. □

4 Hamiltonicity

In this section we show that 8-TD contains a bottleneck Hamiltonian cycle. For some point sets, 5-TD does not contain any bottleneck Hamiltonian cycle.

Theorem 4. *For every point set P in general position,* 8*-TD has a bottleneck Hamiltonian cycle.*

Proof. Let $\mathcal{H}$ be the set of all Hamiltonian cycles through the points of P. Define a total order on the elements of $\mathcal{H}$ by their weight sequence. If two elements have exactly the same weight sequence, break ties arbitrarily to get a total order. Let $H^* = a_0, a_1, \ldots, a_{n-1}, a_0$ be a cycle in $\mathcal{H}$ with minimal weight sequence. It is obvious that H^* is a bottleneck Hamiltonian cycle of P. We will show that all the edges of H^* are in 8-TD. Consider any edge $e = (a_i, a_{i+1})$ in H^* and let $t(a_i, a_{i+1})$ be the triangle corresponding to e (all the index manipulations are modulo n).

Claim 1: None of the edges of H^* can be completely in the interior $t(a_i, a_{i+1})$. Suppose there is an edge $f = (a_j, a_{j+1})$ inside $t(a_i, a_{i+1})$. Let H be a cycle obtained from H^* by deleting e and f, and adding (a_i, a_j) and (a_{i+1}, a_{j+1}). By Observation 2, $t(a_i, a_{i+1}) \succ \max\{t(a_i, a_j), t(a_{i+1}, a_{j+1})\}$, and hence $\mathrm{WS}(H) \prec \mathrm{WS}(H^*)$. This contradicts the minimality of H^*.

Therefore, we may assume that no edge of H^* lies completely inside $t(a_i, a_{i+1})$. Suppose there are w points of P inside $t(a_i, a_{i+1})$. Let $U = u_1, u_2, \ldots, u_w$ represent these points indexed in the order we would encounter them on H^* starting from a_i. Let $R' = r_1, r_2, \ldots, r_w$ represent the vertices where r_i is the vertex succeeding u_i in the cycle. All the vertices in R', probably except r_w, are different from a_i and a_{i+1}. Let $R = R' - \{r_w\}$. Without loss of generality assume that $a_i \in C^4_{a_{i+1}}$, and $t(a_i, a_{i+1})$ is anchored at a_{i+1}, as shown in Figure 4.

Claim 2: For each $r_j \in R$, $t(r_j, a_{i+1}) \succeq \max\{t(a_i, a_{i+1}), t(u_j, r_j)\}$. Suppose there is a point $r_j \in R$ such that $t(r_j, a_{i+1}) \prec \max\{t(a_i, a_{i+1}), t(u_j, r_j)\}$. Construct a new cycle H by removing the edges (u_j, r_j), (a_i, a_{i+1}) and adding the edges (a_{i+1}, r_j) and (a_i, u_j). Since the two new edges have length strictly less than $\max\{t(a_i, a_{i+1}), t(u_j, r_j)\}$, $\mathrm{WS}(H) \prec \mathrm{WS}(H^*)$; which is a contradiction.

Claim 3: For each $r_j, r_k \in R$, $t(r_j, r_k) \succeq \max\{t(a_i, a_{i+1}), t(u_j, r_j), t(u_k, r_k)\}$. Suppose there is a pair r_j and r_k such that $t(r_j, r_k) \prec \max\{t(a_i, a_{i+1}), t(u_j, r_j), d(u_k, r_k)\}$. Construct a cycle H from H^* by first deleting (u_j, r_j), (u_k, r_k), (a_i, a_{i+1}). This results in three paths. One of the paths must contain both a_i and either r_j or r_k. W.l.o.g. suppose that a_i and r_k are on the same path. Add the edges (a_i, u_j), (a_{i+1}, u_k), (r_j, r_k). Since $\max\{t(u_j, r_j), t(u_k, r_k), d(a_i, a_{i+1})\} \succ \max\{t(a_i, u_j), t(a_{i+1}, u_k), t(r_j, r_k)\}$, $\mathrm{WS}(H) \prec \mathrm{WS}(H^*)$; we get a contradiction.

We use Claim 2 and Claim 3 to show that the size of R is at most seven, and consequently $w \leq 8$. Consider the lines $l^0_{a_{i+1}}$, $l^{60}_{a_{i+1}}$, $l^{120}_{a_{i+1}}$, and $l^{120}_{a_i}$ as shown in Figure 4. Let l_1 and l_2 be the rays starting at the corners of $t(a_i, a_{i+1})$ opposite to a_{i+1} and parallel to $l^0_{a_{i+1}}$ and $l^{60}_{a_{i+1}}$ respectively. These lines and rays partition the plane into 12 regions, as shown in Figure 4. We will show that each of the regions D_1, D_2, D_3, D_4, C_1, C_2, and $B = B_1 \cup B_2$ contains at most one point of R, and the other regions do not contain any point of R. Consider the hexagon $X(a_{i+1}, a_i)$. By Claim 2 and Observation 4, no point of R can be inside $X(a_{i+1}, a_i)$. Moreover, no point of R can be inside the cones A_1, A_2, or A_3, because if $r_j \in \{A_1 \cup A_2 \cup A_3\}$, the (upward) triangle $t'(u_j, r_j)$ contains a_{i+1}. Then by Observation 4, $t(r_j, a_{i+1}) \prec t(u_j, r_j)$; which contradicts Claim 2.

We show that each of the regions D_1, D_2, D_3, D_4 contains at most one point of R. Consider the region D_1; by similar reasoning we can prove the claim for D_2, D_3, D_4. Using contradiction, let r_j and r_k be two points in D_1, and w.l.o.g. assume that r_j is the farthest to $l^{60}_{a_{i+1}}$. Then r_k can lie inside any of the cones $C^1_{r_j}$, $C^5_{r_j}$, and $C^6_{r_j}$ (but not in X). If $r_k \in C^1_{r_j}$, then $t'(r_j, r_k)$ is smaller than $t'(a_i, a_{i+1})$ which means that $t(r_j, r_k) \prec t(a_i, a_{i+1})$. If $r_k \in C^5_{r_j}$, then $t'(u_j, r_j)$ contains r_k, that is $t(r_j, r_k) \prec t(u_j, r_j)$. If $r_k \in C^6_{r_j}$, then $t(u_j, r_j)$ contains r_k, that is $t(r_j, r_k) \prec t(u_j, r_j)$. All cases contradict Claim 3.

Now consider the region C_1 (or C_2). By contradiction assume that it contains two points r_j and r_k. Let r_j be the farthest from $l^0_{a_{i+1}}$. It is obvious that $t'(u_j, r_j)$ contains r_k, that is $t(r_j, r_k) \prec t(u_j, r_j)$; which contradicts Claim 3.

Consider the region $B = B_1 \cup B_2$. If both r_j and r_k belong to B_2, then $t'(r_j, r_k)$ is smaller that $t(a_i, a_{i+1})$. If $r_j \in B_1$ and $r_k \in B_2$, then $t'(u_j, r_j)$ contains r_k, and hence $t(r_j, r_k) \prec t(u_j, r_j)$. If both r_j and r_k belong to B_1, let r_j be the farthest from $l^{120}_{a_i}$. Clearly, $t(u_j, r_j)$ contains r_k and hence $t(r_j, r_k) \prec t(u_j, r_j)$. All cases contradict Claim 3.

Therefore, any of the regions D_1, D_2, D_3, D_4, C_1, C_2, and $B = B_1 \cup B_2$ contains at most one point of R. Thus, $|R| \leq 7$ and $w \leq 8$, and $t(a_i, a_{i+1})$ contains at most 8 points of P. Therefore, $e = (a_i, a_{i+1})$ is an edge of 8-TD. □

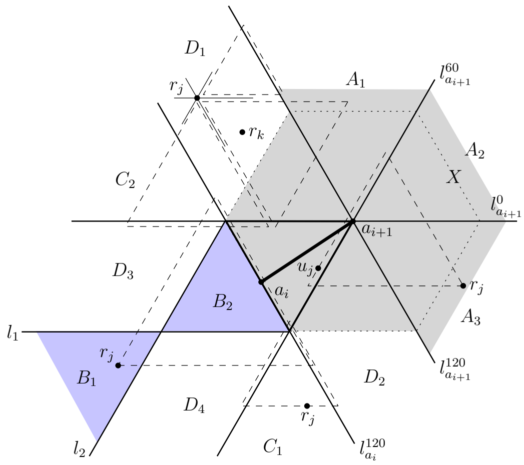


Fig. 4. Illustration of Theorem 4.

5 Perfect Matching Admissibility

In this section we consider the matching problem in k-TD graphs.

Theorem 5. *For a set P of an even number of points in general position in the plane, 6-TD contains a bottleneck perfect matching.*

For some point sets, 5-TD does not contain any bottleneck perfect matching. As for the maximum matching, in [4] the authors proved a tight lower bound of $\lceil \frac{n-1}{3} \rceil$ on the size of a maximum matching in 0-TD. We prove that 1-TD has a matching of size at least $\lceil \frac{2(n-1)}{5} \rceil$ and 2-TD has a matching of size $\lceil \frac{n-1}{2} \rceil$.

Let $\mathcal{P} = \{P_1, P_2, \dots\}$ be a partition of the points in P. Let $G(\mathcal{P})$ be the complete graph with vertex set $\mathcal{P}$. For each edge $e = (P_i, P_j)$ in $G(\mathcal{P})$, let $w(e)$ be equal to the area of the smallest triangle between a point in P_i and a point in P_j, i.e. $w(e) = \min\{t(a, b) : a \in P_i, b \in P_j\}$. That is, the weight of an edge $e \in G(\mathcal{P})$ corresponds to the size of the smallest triangle $t(e)$ defined by the endpoints of e. Let $\mathcal{T}$ be a minimum spanning tree of $G(\mathcal{P})$. Let T be the set of triangles corresponding to the edges of $\mathcal{T}$, i.e. $T = \{t(e) : e \in \mathcal{T}\}$.

Lemma 1. *The interior of any triangle in T does not contain any point of P.*

Lemma 2. *Each point in the plane can be in the interior of at most three triangles in T.*

The following two theorems are based on Lemma 1, Lemma 2, and Theorem 1.

Theorem 6. *For every set P of n points in general position in the plane, 2-TD has a matching of size $\lceil \frac{n-1}{2} \rceil$.*

Proof. First we show that by removing a set K of k points from 2-TD, at most $k+1$ components are generated. Let K be a set of k vertices removed from 2-TD, and let $\mathcal{C} = \{C_1, \ldots, C_{m(k)}\}$ be the resulting $m(k)$ components, where m is a function depending on k. Actually, $\mathcal{C} = \text{2-TD} - K$ and $\mathcal{P} = \{V(C_1), \ldots, V(C_{m(k)})\}$ is a partition of the vertices in $P \setminus K$.

Claim 1. $m(k) \leq k+1$. Let $G(\mathcal{P})$ be a complete graph with vertex set $\mathcal{P}$ which is constructed as described above. Let $\mathcal{T}$ be a minimum spanning tree of $G(\mathcal{P})$ and let T be the set of triangles corresponding to the edges of $\mathcal{T}$. It is obvious that $\mathcal{T}$ contains $m(k)-1$ edges and hence $|T| = m(k)-1$. Let $F = \{(p,t) : p \in K, t \in T, p \in t\}$ be the set of all (point, triangle) pairs where $p \in K$, $t \in T$, and p is inside t. By Lemma 2 each point in K can be inside at most three triangles in T. Thus, $|F| \leq 3 \cdot |K|$. Now we show that each triangle in T contains at least three points of K. Consider any triangle $\tau \in T$. Let $e = (V(C_i), V(C_j))$ be the edge of $\mathcal{T}$ which is corresponding to τ, and let $a \in V(C_i)$ and $b \in V(C_j)$ be the points defining τ. By Lemma 1, τ does not contain any point of $P \setminus K$ in its interior. Therefore, τ contains at least three points of K, because otherwise (a,b) is an edge in 2-TD which contradicts the fact that a and b belong to different components in $\mathcal{C}$. Thus, each triangle in T contains at least three points of K in its interior. That is, $3 \cdot |T| \leq |F|$. Therefore, $3(m(k)-1) \leq |F| \leq 3k$, and hence $m(k) \leq k+1$.

Note that $o(\mathcal{C}) \leq |C| = m(k)$. By Claim 1, $m(k) \leq k+1$. Thus, $o(\mathcal{C}) \leq k+1$. This implies that def(2-TD) ≤ 1. Therefore, by Theorem 1, the size of a maximum matching, M^*, is $\frac{n-1}{2}$. Since $|M^*|$ is an integer number, $|M^*| = \lceil \frac{n-1}{2} \rceil$. □

Theorem 7. *For every set P of n points in general position in the plane, 1-TD has a matching of size at least $\lceil \frac{2(n-1)}{5} \rceil$.*

Proof. Let K be a set of k vertices removed from 1-TD, and let $\mathcal{C} = \{C_1, \ldots, C_{m(k)}\}$ be the resulting $m(k)$ components. Actually, $\mathcal{C} = \text{1-TD} - K$ and $\mathcal{P} = \{V(C_1), \ldots, V(C_{m(k)})\}$ is a partition of the vertices in $P \setminus K$. Note that $o(\mathcal{C}) \leq m(k)$. Let M^* be a maximum matching in 1-TD. By Theorem 1,

$$|M^*| = \frac{1}{2}(n - \text{def(1-TD)}), \tag{1}$$

where

$$\text{def(1-TD)} = \max_{K \subseteq P}(o(\mathcal{C}) - |K|) \leq \max_{K \subseteq P}(|\mathcal{C}| - |K|) = \max_{0 \leq k \leq n}(m(k) - k). \tag{2}$$

Define $G(\mathcal{P})$, $\mathcal{T}$, T, and F as in the proof of Theorem 6. By Lemma 2, $|F| \leq 3 \cdot |K|$. By the same reasoning as in the proof of Theorem 6, each triangle in

T has at least two points of K in its interior. Thus, $2 \cdot |T| \leq |F|$. Therefore, $2(m(k) - 1) \leq |F| \leq 3k$, and hence

$$m(k) \leq \frac{3k}{2} + 1. \tag{3}$$

In addition, $k + m(k) = |K| + |\mathcal{C}| \leq |P| = n$, and hence

$$m(k) \leq n - k. \tag{4}$$

By Inequalities (3) and (4),

$$m(k) \leq \min\{\frac{3k}{2} + 1, n - k\}. \tag{5}$$

Thus, by (2) and (5)

$$\begin{aligned} \text{def(1-TD)} &\leq \max_{0 \leq k \leq n} (m(k) - k) \\ &\leq \max_{0 \leq k \leq n} \{\min\{\frac{3k}{2} + 1, n - k\} - k\} \\ &= \max_{0 \leq k \leq n} \{\min\{\frac{k}{2} + 1, n - 2k\}\} = \frac{n+4}{5}, \end{aligned} \tag{6}$$

where the last equation is obtained by setting $\frac{k}{2} + 1$ equal to $n - 2k$. Finally by substituting (6) in Equation (1) we have $|M^*| \geq \frac{2(n-1)}{5}$. Sine $|M^*|$ is an integer number, $|M^*| \geq \lceil \frac{2(n-1)}{5} \rceil$. □

6 Blocking TD-Delaunay graphs

In this section we consider the problem of blocking TD-Delaunay graphs. Let P be a set of n points in general position in the plane. Recall that a point set K blocks k-TD(P) if in k-TD($P \cup K$) there is no edge connecting two points in P. That is, P is an independent set in k-TD($P \cup K$).

Theorem 8. *At least $\lceil \frac{(k+1)(n-1)}{3} \rceil$ points are necessary to block k-TD(P).*

Proof. Let K be a set of m points which blocks k-TD(P). Let $G(\mathcal{P})$ be the complete graph with vertex set $\mathcal{P} = P$. Let $\mathcal{T}$ be a minimum spanning tree of $G(\mathcal{P})$ and let T be the set of triangles corresponding to the edges of $\mathcal{T}$. It is obvious that $|T| = n - 1$. By Lemma 1 the triangles in T are empty, thus, the edges of $\mathcal{T}$ belong to any k-TD(P) where $k \geq 0$. To block each edge, corresponding to a triangle in T, at least $k + 1$ points are necessary. By Lemma 2 each point in K can lie in at most three triangles of T. Therefore, $m \geq \lceil \frac{(k+1)(n-1)}{3} \rceil$, which implies that at least $\lceil \frac{(k+1)(n-1)}{3} \rceil$ points are necessary to block all the edges of $\mathcal{T}$ and hence k-TD(P). □

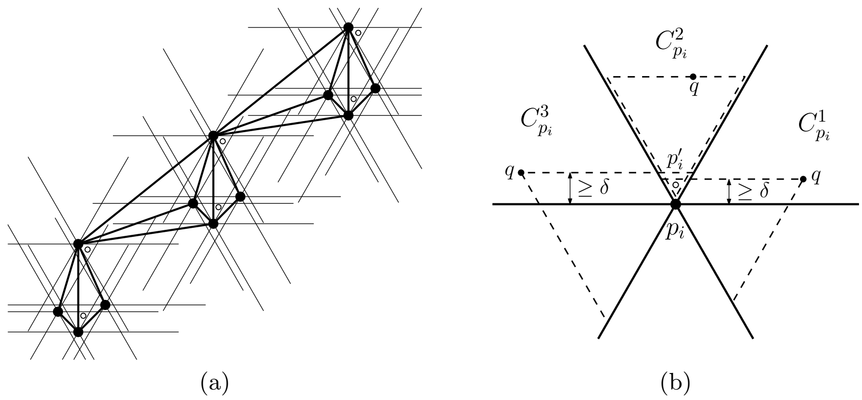


Fig. 5. (a) A 0-TD graph which is shown in bold edges is blocked by $\lceil \frac{n-1}{2} \rceil$ white points, (b) p'_i blocks all the edges connecting p_i to the vertices above $l^0_{p_i}$.

By Theorem 8, at least $\lceil \frac{n-1}{3} \rceil$, $\lceil \frac{2(n-1)}{3} \rceil$, $n-1$ points are necessary to block 0-, 1-, 2-TD(P) respectively. Now we introduce another formula which gives a better lower bound for 0-TD. For a point set P, let $\nu_k(P)$ and $\alpha_k(P)$ respectively denote the size of a maximum matching and a maximum independent set in k-TD(P). For every edge in the maximum matching, at most one of its endpoints can be in the maximum independent set. Thus,

$$\alpha_k(P) \leq |P| - \nu_k(P). \tag{7}$$

Let K be a set of m points which blocks k-TD(P). By definition there is no edge between points of P in k-TD($P \cup K$). That is, P is an independent set in k-TD($P \cup K$). Thus,

$$n \leq \alpha_k(P \cup K). \tag{8}$$

By (7) and (8) we have

$$n \leq \alpha_k(P \cup K) \leq (n+m) - \nu_k(P \cup K). \tag{9}$$

Theorem 9. *At least $\lceil \frac{n-1}{2} \rceil$ points are necessary to block 0-TD(P).*

Proof. Let K be a set of m points which blocks 0-TD(P). Consider 0-TD($P \cup K$). It is known that $\nu_0(P \cup K) \geq \lceil \frac{n+m-1}{3} \rceil$; see [4]. By Inequality (9),

$$n \leq (n+m) - \lceil \frac{n+m-1}{3} \rceil \leq \frac{2(n+m)+1}{3},$$

and consequently $m \geq \lceil \frac{n-1}{2} \rceil$ (note that m is an integer number). □

Figure 5(a) shows a 0-TD graph on a set of 12 points which is blocked by 6 points. By removing the topmost point we obtain a set with odd number of points which can be blocked by 5 points.

Theorem 10. *There exists a set K of $(k+1)(n-1)$ points that blocks k-TD(P).*

This bound is tight. Consider the case where $k = 0$. In this case 0-TD(P) can be a path representing $n - 1$ disjoint triangles and for each triangle we need at least one point to block its corresponding edge. In k-TD(P) we need at least $k + 1$ points to block each of these edges.

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