Acute Tours in the Plane

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Abstract

We confirm the following conjecture of Fekete and Woeginger from 1997: for any sufficiently large even number n, every set of n points in the plane can be connected by a spanning tour (Hamiltonian cycle) consisting of straight-line edges such that the angle between any two consecutive edges is at most $\pi/2$. Our proof is constructive and suggests a simple $O(n \log n)$-time algorithm for finding such a tour. The previous best-known upper bound on the angle is $2\pi/3$, and it is due to Dumitrescu, Pach and Tóth (2009).

1 Introduction

The Euclidean traveling salesperson problem (TSP) is a well-studied and fundamental problem in combinatorial optimization and computational geometry. In this problem we are given a set of points in the plane and our goal is to find a shortest tour that visits all points. Motivated by applications in robotics and motion planning, in recent years there has been an increased interest in the study of tours with bounded angles at vertices, rather than bounded length of edges; see e.g. [2, 3, 13, 14, 15] and references therein. Bounded-angle structures (tours, paths, trees) are also desirable in the context of designing networks with directional antennas [6, 7, 11, 19]. Bounded-angle tours (and paths), in particular, have received considerable attention following the PhD thesis of S. Fekete [14] and the seminal work of Fekete and Woeginger [15].

Consider a set P of at least three points in the plane. A *spanning tour* is a directed Hamiltonian cycle on P that is drawn with straight-line edges. When three consecutive vertices p_i, p_{i+1}, p_{i+2} of the tour are traversed in this order, the *rotation angle* at p_{i+1} (denoted by $\angle p_i p_{i+1} p_{i+2}$) is the angle in $[0, \pi]$ that is determined by the segments $p_i p_{i+1}$ and $p_{i+1} p_{i+2}$. If all rotation angles in a tour are at most $\pi/2$ then it is called an *acute* tour.

In 1997, Fekete and Woeginger [15] raised many challenging questions about bounded-angle tours and paths. In particular they conjectured that *for any sufficiently large even number n, every set of n points in the plane admits an acute spanning tour* (*a tour with rotation angles at most* $\pi/2$). They stated the conjecture specifically for $n \geqslant 8$. The point set illustrated in Figure 1(a) (also described in [15]) shows that the upper bound $\pi/2$ is the best achievable. The conjecture does not hold if n is allowed to be an odd number; for example if the n points are on a line then in any spanning tour one of the rotation angles must be π. The conjecture also does not hold if n is allowed to be small. For instance the 4-element point set consisting of the 3 vertices of an equilateral triangle with its center, must have a rotation angle $2\pi/3$ in any spanning tour. Also the 6-element point set of Figure 1(b) (also illustrated in [15] and [13]) must have a rotation angle of at least $2\pi/3 - \epsilon$ in any spanning tour, for some arbitrary small constant ϵ.

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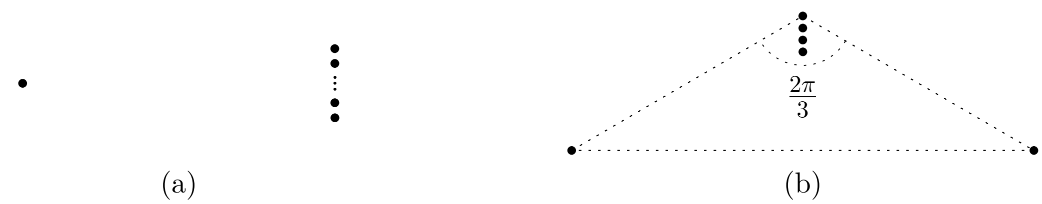


Figure 1: (a) a general lower bound example, and (b) a lower bound example for 6 points.

In 2009, Dumitrescu, Pach and Tóth [13] took the first promising steps towards proving the conjecture. They confirmed the conjecture for points in convex position. For general point sets, they obtained the first partial result by showing that any even size point set admits a spanning tour in which each rotation angle is at most $2\pi/3$.

In this paper we prove the conjecture of Fekete and Woeginger for general point sets.

Theorem 1. *Let $n \geqslant 20$ be an even integer. Then every set of n points in the plane admits an acute spanning tour. Such a tour can be computed in linear time after finding an equitable partitioning of the points with two orthogonal lines.*

Due to our desire of having a short proof, we prove the conjecture for $n \geqslant 20$. Perhaps with some detailed case analysis one could extend the range of n to a number smaller than 20.

Difficulties towards a proof. Fekete and Woeginger [15] exhibited an arbitrary-large even-size point set for which an algorithm (or a proof technique), that always outputs the longest tour or includes the diameter in the solution, does not achieve an acute tour; the point set is similar to that of Figure 1(b) but has more than 6 points. This somehow breaks the hope for finding an acute tour by using greedy techniques. Therefore, to prove the conjecture one might need to employ some nontrivial ideas.

1.1 Related problems

Another interesting conjecture of Fekete and Woeginger [15] is that any set of points in the plane admits a spanning path in which all rotation angles are at least $\pi/6$.[1] In 2008, Bárány, Pór, and Valtr [8] obtained the first constant lower bound of $\pi/9$, thereby gave a partial answer to the conjecture. The full conjecture was then proved, although not yet written in a paper format, by J. Kynčl [16] (see also the note added in the proof of [8]).

Fekete and Woeginger [15] showed that any set of points in the plane admits an acute spanning path (where all intermediate rotation angles are at most $\pi/2$). Such a path can be obtained simply by starting from an arbitrary point and iteratively connecting the current point to its farthest among the remaining points. Notice that the resulting path always contains the diameter and by the difficulties mentioned above it cannot be completed to an acute tour. Carmi et al. [11] showed how to construct acute paths with shorter edges; again no guarantee to be completed to an acute tour. Aichholzer et al. [4] studied a similar problem with an additional constraint that the path should be *plane* (i.e., its edges do not cross each other). Among other results, they showed that any set of points in the plane in general position admits a plane spanning path with rotation angles at most $3\pi/4$. They also conjectured that this upper bound could be replaced by $\pi/2$.

[1] This bound is the best achievable as the three vertices of an equilateral triangle together with its center do not admit a path with rotation angles greater than $\pi/6$.

The bounded-angle minimum spanning tree (also known as α-MST) is a related problem that asks for a Euclidean minimum spanning tree in which all edges incident to every vertex lie in a cone of angle at most α. This problem is motivated by replacing omni-directional antennas—in a wireless network—with directional antennas, which are more secure, require lower transmission ranges, and cause less interference; see e.g. [6, 7, 9, 10, 19].

Another related problem (with an objective somewhat opposite to ours) is to minimize the total *turning angle* of the tour [2].[2] Similar problems are also studied under the concepts of *pseudo-convex* tours and paths (which make only right turns) [15], and *reflexivity* of a point set (which is the smallest number of reflex vertices in a simple polygonalization of the point set) [1, 5].

The so-called *Tverberg cycle* is a cycle with straight-line edges such that the diametral disks[3] induced by the edges have nonempty intersection. Recently, Pirahmad et al. [17] showed how to construct a spanning Tverberg cycle on any set of points in the plane. Although the constructed cycle has many acute angles, it is still far from being fully acute.

Remark. It is worth mentioning that having a tour with many acute angles does not necessarily help in getting a fully acute tour because one can simply get a tour with at least $n-2$ acute angles by interconnecting the endpoints of acute paths obtained in [11, 15].

2 Preliminaries for the proof

A set of four points in the plane is called a *quadruple*. If the four points are in convex position then the quadruple is called *convex*, otherwise it is called *concave*; the quadruple in Figure 2(a) is convex while the quadruples in Figures 2(b) and 2(c) are concave. We refer to the interior point of a concave quadruple as its *center*. By connecting the center of a concave quadruple to its other three points we obtain three angles. If one of these angles is at most $\pi/2$ then the quadruple is called *concave-acute*, otherwise all the angles are larger than $\pi/2$ and the quadruple is called *concave-obtuse*; the quadruple in Figure 2(b) is concave-acute while the one in Figure 2(c) is concave-obtuse.

A path, that is drawn by straight-line edges, is called *acute* if all the angles determined by its adjacent edges are at most $\pi/2$. For two directed paths P_1 and P_2, where P_1 ends at the same vertex at which P_2 starts, we denote their concatenation by $P_1 \oplus P_2$.

For two distinct points p and q in the plane, we say that p is *to the left of* q if the x-coordinate of p is not larger than the x-coordinate of q. Analogously, we say that p is *below* q if the y-coordinate of p is not larger than the y-coordinate of q.

It is known that any set of n points in the plane can be split into four parts of equal size using two orthogonal lines (see e.g. [18] or [12, Section 6.6]); such two lines can be computed in $\Theta(n \log n)$ time [18]. The following is a restatement of this result which is borrowed from [13].

Lemma 1. *Given a set S of n points in the plane (n even), one can always find two orthogonal lines ℓ_1, ℓ_2 and a partition $S = S_1 \cup S_2 \cup S_3 \cup S_4$ with $|S_1| = |S_3| = \lfloor \frac{n}{4} \rfloor$ and $|S_2| = |S4| = \lceil \frac{n}{4} \rceil$ such that S_1 and S_3 belong to two opposite closed quadrants determined by ℓ_1 and ℓ_2, and S_2 and S_4 belong to the other two opposite closed quadrants.*

Our proof of Theorem 1 shares some similarities with that of Dumitrescu et al. [13] (for points in convex position) in the sense that both proofs employ the equitable partitioning of Lemma 1. However, there are major differences between the two proofs mainly because simple structures, that

[2]The turning angle at a vertex v is the change in the direction of motion at v when traveling on the tour. It is essentially π minus the rotation angle at v.

[3]The diametral disk induced by an edge pq is the disk that has pq as its diameter.

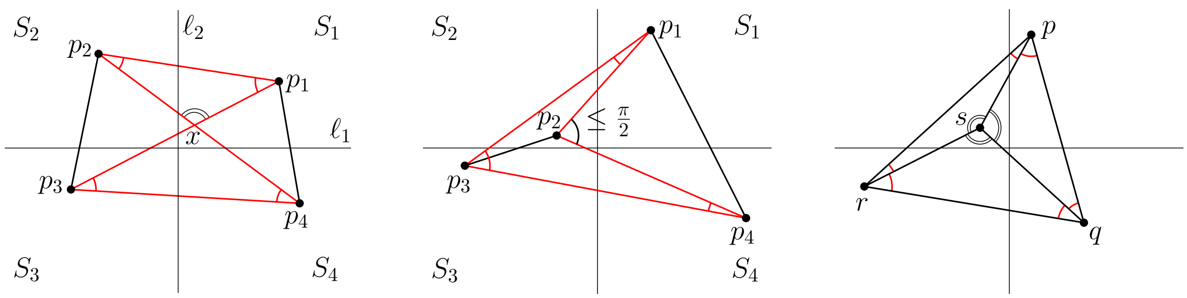
appear in points in convex position, do not necessarily appear in general point sets. Therefore one needs to extract complex structures from general point sets and combine them to establish a proof.

3 Proof of Theorem 1

Throughout this section we assume that n is an even integer. We show how to construct an acute tour on any set of $n \geqslant 20$ points in the plane, and thereby prove Theorem 1. In Subsection 3.1 we describe the setup for our construction, and then in Subsection 3.2 we construct the tour.

3.1 The proof setup

Let S be a set of $n \geqslant 20$ points in the plane. Let $\{S_1, S_2, S_3, S_4\}$ be an equitable partitioning of S with two orthogonal lines ℓ_1 and ℓ_2 that satisfies the conditions of Lemma 1. After a suitable rotation and translation we may assume that ℓ_1 and ℓ_2 coincide with the x and y coordinate axes, respectively. Also, after a suitable relabeling we may assume that all points of S_i belong the ith quadrant determined by the axes as depicted in Figure 2(a).



(a) Convex quadruple (b) Concave-acute quadruple (c) Concave-obtuse quadruple

Figure 2: Illustration of (a) Lemma 2 where P is convex and $\angle p_1xp_2 \geqslant \pi/2$, (b) Lemma 2 where P is concave-acute and $\angle p_1p_2p_4 \leqslant \pi/2$, and (c) Lemma 3 where all the three angles at s are obtuse.

Based on the above partitioning we introduce four types of quadruples. Let $P = \{p_1, p_2, p_3, p_4\}$ be a quadruple such that $p_i \in S_i$ for all $i = 1, 2, 3, 4$. We say that P is *upward* if the path $p_2p_4p_3p_1$ (or equivalently $p_1p_3p_4p_2$) is acute, *downward* if the path $p_3p_1p_2p_4$ (or equivalently $p_4p_2p_1p_3$) is acute, *leftward* if the path $p_2p_4p_1p_3$ (or equivalently $p_3p_1p_4p_2$) is acute, and *rightward* if the path $p_1p_3p_2p_4$ (or equivalently $p_4p_2p_3p_1$) is acute. Such paths are referred to as "hooks" in [13]. The following lemmas and observation, although very simple, play important roles in our proof.

Lemma 2. *Let $P = \{p_1, p_2, p_3, p_4\}$ be a quadruple such that $p_i \in S_i$ for all $i = 1, 2, 3, 4$. If P is convex or concave-acute then it is upward and downward or it is leftward and rightward.*

Proof. First assume that P is convex. Let x denote the intersection point of the diagonals p_1p_3 and p_2p_4. If $\angle p_1xp_2 \geqslant \pi/2$ then the paths $p_2p_4p_3p_1$ and $p_3p_1p_2p_4$ are acute and thus P is upward and downward; see Figure 2(a). If $\angle p_1xp_4 \geqslant \pi/2$ then the paths $p_2p_4p_1p_3$ and $p_1p_3p_2p_4$ are acute and thus P is leftward and rightward.

Now assume that P is concave-acute. Without loss of generality we assume that p_2 is the center of P. Observe that in this case $\angle p_1p_2p_3$ is obtuse. This and the fact that P is concave-acute imply that one of $\angle p_1p_2p_4$ and $\angle p_3p_2p_4$ is acute. If $\angle p_1p_2p_4$ is acute as depicted in Figure 2(b) then the paths $p_2p_4p_3p_1$ and $p_3p_1p_2p_4$ are acute and thus P is upward and downward (observe that $\angle p_2p_1p_3 + \angle p_1p_3p_4 + \angle p_3p_4p_2 = \angle p_1p_2p_4 \leqslant \pi/2$). Analogously, if $\angle p_3p_2p_4$ is acute then the paths $p_2p_4p_1p_3$ and $p_1p_3p_2p_4$ are acute and thus P is leftward and rightward. □

Lemma 3. *Let $\{p, q, r, s\}$ be a concave-obtuse quadruple with center s. Then all angles $\angle pqs$, $\angle qps$, $\angle qrs$, $\angle rqs$, $\angle rps$, and $\angle prs$ are acute.*

Proof. See Figure 2(c). In each of the triangles $\triangle spq$, $\triangle sqr$, and $\triangle srp$ the angle at s is larger than $\pi/2$. Thus the other two angles are acute. □

Lemma 4. *Let $P = \{p_1, p_2, p_3, p_4\}$ be a quadruple such that $p_i \in S_i$ for all $i = 1, 2, 3, 4$. If P is concave-obtuse then it is upward, downward, leftward, or rightward.*

Proof. Without loss of generality assume that p_2 is the center of P. See Figure 2(c) where $p_2 = s$. In the triangle $\triangle p_1p_3p_4$ the angle at p_1 or the angle at p_3 is acute. If the angle at p_1 is acute then the path $p_2p_4p_1p_3$ is acute and thus P is leftward ($\angle p_2p_4p_1$ is acute by Lemma 3). If the angle at p_3 is acute then the path $p_2p_4p_3p_1$ is acute and thus P is upward ($\angle p_2p_4p_3$ is acute by Lemma 3). □

Observation 1. *Let p, q, and r be any three points in S such that q and r lie in the quadrant that is opposite to the quadrant containing p. Then the angle $\angle qpr$ is acute.*

3.2 The tour construction

In this section we show how to construct an acute tour on S where $|S| \geqslant 20$. By Lemma 1 each S_i with $i \in \{1, 2, 3, 4\}$ has at least $\lfloor 20/4 \rfloor = 5$ points. From each S_i we select an arbitrary subset of 5 points, and then we partition (the total 20) selected points into 5 quadruples such that each quadruple contains exactly one point from each S_i. Let $\mathcal{Q}$ denote the set of these quadruples. For any quadruple X in $\mathcal{Q}$ we denote the points of X by x_1, x_2, x_3, x_4 where $x_i \in S_i$ for all $i = 1, 2, 3, 4$.

Since $|\mathcal{Q}| \geqslant 5$, by the pigeonhole principle $\mathcal{Q}$ has three quadruples that are *vertical* (i.e. upward, downward, or both upward and downward) or three that are *horizontal* (i.e. leftward, rightward, or both leftward and rightward). Without loss of generality assume that Q has three vertical quadruples. If, among these vertical quadruples, we can choose two quadruples of opposite type (i.e., one upward and one downward), then we construct a tour as in case 1 below. Otherwise, the three quadruples are concave-obtuse and of the same type in which case we construct a tour as in case 2 below. Our constructions take linear time in both cases.

Case 1: *$\mathcal{Q}$ contains two quadruples such that one is upward and the other is downward.* Let P and Q be such quadruples where P is upward and Q is downward. Since P is upward, the path $p_1p_3p_4p_2$ is acute. Since Q is downward, the path $q_4q_2q_1q_3$ is acute; see Figure 3. Let $\overline{S_2S_4}$ be a polygonal path starting from p_2, ending in q_4, alternating between S_2 and S_4, and containing all points of $S_2 \cup S_4$ except for q_2 and p_4. Let $\overline{S_3S_1}$ be a polygonal path starting from q_3, ending in p_1, alternating between S_3 and S_1, and containing all points of $S_3 \cup S_1$ except for p_3 and q_1. Such polygonal paths exist because by Lemma 1 we have $|S_2| = |S_4|$ and $|S_1| = |S_3|$. All intermediate angles of these two polygonal paths are acute by Observation 1. Then the tour $p_1p_3p_4p_2 \oplus \overline{S_2S_4} \oplus q_4q_2q_1q_3 \oplus \overline{S_3S_1}$ is acute, and it spans S. Notice that the angles at p_1, p_2, q_3 and q_4 are acute by Observation 1.

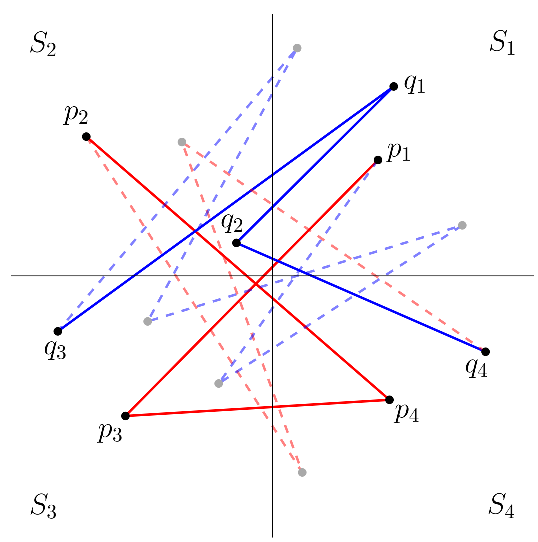


Figure 3: Illustration of Case 1.

Case 2: *$\mathcal{Q}$ contains three concave-obtuse quadruples of the same type.* Let P, Q and R be such quadruples, and without loss of generality assume that they are upward. Thus, the paths $p_2p_4p_3p_1$ and $q_2q_4q_3q_1$ and $r_2r_4r_3r_1$ are acute. Since P, Q and R are concave-obtuse their centers should lie at endpoints of these paths (the centers cannot be interior vertices of acute paths). Thus the center of P is either p_1 or p_2, the center of Q is either q_1 or q_2, and the center of R is either r_1 or r_2. This means that the centers lie in quadrants 1 and 2. By the pigeonhole principle, and after a suitable reflection, we may assume that at least two of the centers lie in quadrant 2. After a suitable relabeling assume that the centers of P and Q (i.e. p_2 and q_2) lie in quadrant 2. The center of R lies either in quadrant 2 (i.e. it is r_2) or in quadrant 1 (i.e. it is r_1).

After a suitable relabeling assume that p_2 lies below q_2, as in Figure 4. Now we build our tour as follows. First we connect p_2 to p_1 and q_1. The point p_2 is below p_1 because p_2 lies below the segment p_1p_3. The point p_2 is also below q_1 because p_2 is below q_2 which is in turn below q_1 (as q_2 lies below the segment q_1q_3). Thus p_2 is below both p_1 and q_1. Also notice that p_2 is to the left of both p_1 and q_1. Thus, the angle $\angle p_1p_2q_1$ is acute (imagine moving the origin to p_2, then both p_1 and q_1 would lie in the first quadrant). Then we connect q_3 to q_1 and q_4. The angle $\angle q_4q_3q_1$ is acute because Q is upward (i.e. the path $q_2q_4q_3q_1$ is acute). The angle $\angle p_2q_1q_3$ is acute because both p_2 and q_3 lie below and to the left of q_1. Therefore, the path $p_1p_2q_1q_3q_4$ is acute; see Figure 4. In the rest of the construction we distinguish two subcases, depending on the center of R.

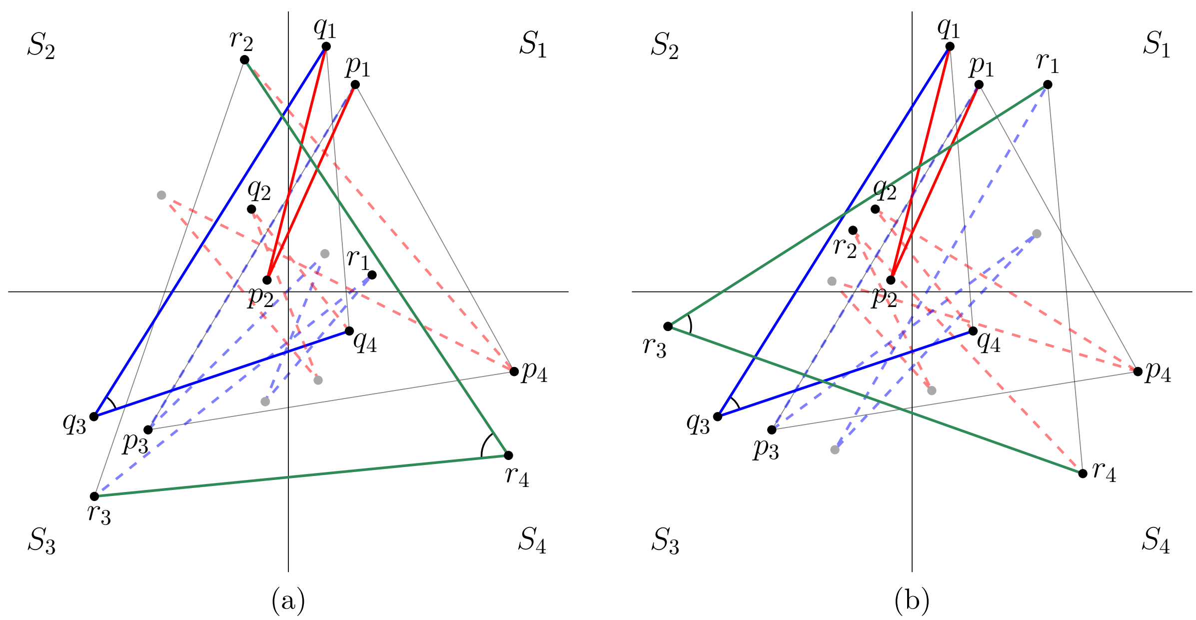


Figure 4: Illustration of Case 2. Three concave-obtuse quadruples P, Q and R that are upward, and the centers of P and Q lie in quadrant 2. (a) Subcase 2.1 where the center of R is in quadrant 1. (b) Subcase 2.2 where the center of R is in quadrant 2.

Subcase 2.1: *The center of R is r_1.* This case is depicted in Figure 4(a). We connect r_4 to r_2 and r_3. The resulting path $r_2r_4r_3$ is acute (because R is upward, i.e., the path $r_2r_4r_3r_1$ is acute). Let $\overline{S_4S_2}$ be a polygonal path starting from q_4, ending in r_2, alternating between S_4 and S_2, containing all points of $S_4 \cup S_2$ except for r_4, p_2, and having q_4q_2 as its first edge. Let $\overline{S_3S_1}$ be a polygonal path starting from r_3, ending in p_1, alternating between S_3 and S_1, containing all points of $S_3 \cup S_1$ except for q_3, q_1, and having r_3r_1 as its first edge and p_3p_1 as its last edge. All intermediate angles of these two paths are acute by Observation 1. By interconnecting the constructed paths we obtain the tour $p_1p_2q_1q_3q_4 \oplus \overline{S_4S_2} \oplus r_2r_4r_3 \oplus \overline{S_3S_1}$ which is acute, and it spans S. The angles at p_1, r_3, q_4 are acute by Lemma 3, and the angle at r_2 is acute by Observation 1.

Subcase 2.2: *The center of R is r_2.* This case is depicted in Figure 4(b). We connect r_3 to r_4 and r_1. The resulting path $r_4r_3r_1$ is acute (because R is upward, i.e. the path $r_2r_4r_3r_1$ is acute). Let $\overline{S_4S_2S_4}$ be a polygonal path starting from q_4, ending in r_4, alternating between S_4 and S_2, containing all points of $S_4 \cup S_2$ except for p_2, and having q_4q_2 as its first edge and r_2r_4 as its last edge. Let $\overline{S_1S_3S_1}$ be a polygonal path starting from r_1, ending in p_1, alternating between S_1 and S_3, containing all points of $S_1 \cup S_3$ except for q_1, q_3, r_3, and having p_3p_1 as its last edge. Intermediate angles of these paths are acute by Observation 1. Thus $p_1p_2q_1q_3q_4 \oplus \overline{S_4S_2S_4} \oplus r_4r_3r_1 \oplus \overline{S_1S_3S_1}$ is an acute spanning tour. The angles at q_4, r_4, and p_1 are acute by Lemma 3, and the angle at r_1 is acute by Observation 1. This finishes our proof of Theorem 1.

4 Concluding remarks

We showed how to construct an acute tour on any set of n points in the plane, where n is even and at least 20. Our construction uses at most 12 points in each case (namely the points of quadruples P, Q and R). One might be interested to extend the range of n (to smaller even numbers) by taking advantage of the 8 unused points, although this may require some case analysis.

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