

COMP2804 Midterm Exam, Winter 2022

First Last 100000000

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This is a closed book exam. You are to do this exam on your own without consulting anyone else or using the internet.

Submit your answers at this URL: <https://forms.gle/1mDZw4etNvKst8CK9>
(Copy and paste the link into your browser if necessary.)

Marking Scheme: Each of the 17 questions is worth 1 mark.

Reminders:

- Binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Newton's Binomial Theorem:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

- Fibonacci numbers:

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2 \end{cases}$$

Question 1 (b). The School of Computer Science has f full professors, a associate professors, t assistant professors, i instructors, and s students. The SCS Executive Committee must consist of

- 2 full (f) professors;
- 1 or 2 associate (a) professor;
- 1 assistant (t) professor;
- 1 instructor (i); and
- 2 students (s).

How many ways are there to form an SCS Executive Committee?

- ✗ $\binom{f}{2} \cdot \binom{a}{2} \cdot t \cdot i \cdot \binom{s}{2}$
- ✗ $\binom{f}{2} \cdot a \cdot t \cdot i \cdot \binom{s}{2}$
- ✗ $\binom{f}{2} \cdot a^2 \cdot t \cdot i \cdot \binom{s}{2}$
- ✗ $\binom{f}{2} \cdot 2a \cdot t \cdot i \cdot \binom{s}{2}$
- ✓ $\binom{f}{2} \cdot a \cdot t \cdot i \cdot \binom{s}{2} + \binom{f}{2} \cdot \binom{a}{2} \cdot t \cdot i \cdot \binom{s}{2}$

Question 2 (b). A class contains $n \geq 2$ distinct students and wants to send a group of $k \leq n - 2$ students on a field trip. Two of these students, Fred and Steve, are bullies. The rest of the students are not bullies.

How many ways are there to choose a set of k students that does not contain two bullies?

- ✗ $\binom{n-1}{k} + \binom{n-2}{k}$
- ✗ $\binom{n-2}{k}$
- ✗ $\binom{n-2}{k-1} \cdot (n - k - 1)$
- ✗ $\binom{n-1}{k}$
- ✓ $\binom{n-2}{k} + 2 \cdot \binom{n-2}{k-1}$

Question 3 (b). Piper (P) and Marley (M) are getting married. In addition to Piper and Marley, the wedding party has p of Piper's friends P_1, \dots, P_p and m of Marley's friends M_1, \dots, M_m . Piper and Marley have no common friends, so $\{P_1, \dots, P_p\} \cap \{M_1, \dots, M_m\} = \emptyset$.

It's later in the evening, everyone has been drinking, and Piper and Marley had a fight. It's time to take a wedding photo and we want to line up the entire wedding party without having Piper and Marley standing next to each other. For example we could line them up like $M, P_1, \dots, P_p, P, M_1, \dots, M_m$.

How many ways are there to line up to the group so that Piper and Marley are *not* standing beside each other?

- ✗ $p!m!$
- ✗ $2p!m!$
- ✗ $4p!m!$
- ✗ $p!m! + p!m!$
- ✓ $(p + m + 1)!(p + m)$

Question 4 (b). Let's go to the animal shelter to take home some cats. At the shelter there are $b \geq 5$ black cats B_1, \dots, B_b and $w \geq 5$ white cats W_1, \dots, W_w . All the cats are distinct; we can distinguish between any two cats, even if they have the same colour.

How many ways are there to take home 5 cats so that we take home an odd number of white cats?

- $\binom{b+w}{5}$
- $\binom{b}{5} + \binom{b}{3} \cdot \binom{w}{2} + \binom{b}{1} \cdot \binom{w}{4}$
- $\binom{w}{5} + \binom{w}{3} \cdot \binom{b}{2} + \binom{w}{1} \cdot \binom{b}{4}$
- $\binom{w}{5} + \binom{w}{4} \cdot b + \binom{w}{3} \cdot \binom{b}{2}$
- $\sum_{k=0}^5 \binom{w}{k} \cdot \binom{b}{5-k}$

Question 5 (b). Let $n \geq 5$. How many strings of length n over the alphabet $\{a, b, c\}$ begin with abc or end with bb ?

- 3^n
- 3^{n-5}
- $3^{n-2} - 3^{n-5}$
- $3^{n-3} + 3^{n-2} - 3^{n-5}$
- $3^n - 3^{n/2}$

Question 6 (b). How many strings can be obtained by rearranging the letters of the word SIMSANTEATERS

- $13!$
- $13!/48$
- $13 \cdot 12 \cdot \binom{11}{3} \cdot \binom{9}{2} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot 2 \cdot 1$
- $13 \cdot 12 \cdot \binom{11}{2} \cdot \binom{9}{3} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot 2 \cdot 1$
- $13 \cdot 12 \cdot \binom{11}{2} \cdot \binom{9}{2} \cdot \binom{7}{3} \cdot \binom{5}{3} \cdot 2 \cdot 1$

Question 7 (b). What does $\sum_{k=1}^n \binom{n}{k} 2^{n-k}$ count?

- The number of strings of length n over the alphabet $\{a, b\}$
- The number of strings of length n over the alphabet $\{a, b\}$ that contain at least one a
- The number of strings of length n over the alphabet $\{a, b, c\}$
- The number of strings of length n over the alphabet $\{a, b, c\}$ that contain no a
- The number of strings of length n over the alphabet $\{a, b, c\}$ that contain at least one a

Question 8 (b). I have a jar with 55 balls numbered $1, \dots, 55$. I want to take balls out of the jar until I find two different pairs of balls $\{b_1, b_2\}$ and $\{b_3, b_4\}$ such that $b_1 - b_2 = b_3 - b_4$.

The fewest balls I must take out before I am guaranteed this will happen is:

- 4 balls
- 11 balls
- 15 balls
- 27 balls
- 55 balls

Question 9 (b). What is the coefficient of $x^{13}y^9$ in the expansion of $(3x - 2y)^{22}$?

- $\binom{22}{9} \cdot 3^{13} \cdot 2^9$
- $-\binom{22}{11} \cdot 3^{13} \cdot 2^9$
- $\binom{22}{12} \cdot 3^{13} \cdot 2^9$
- $-\binom{22}{13} \cdot 3^{13} \cdot 2^9$
- None of the other answers is correct

Question 10 (b). A string over the alphabet $\{a, b, c\}$ is called *great* if it does not contain bc or ba . Let $n \geq 2$. How many great strings of length n are there?

- 2^n
- $2^{n+1} - 1$
- 3^n
- $3^n - 2^n$
- f_{n+1}

Question 11 (b). A bitstring is called 00-free if it does not contain two 0s next to each other. In class we have seen that, for any $m \geq 1$, the number of 00-free bitstrings of length m is equal to the $(m + 2)$ th Fibonacci number f_{m+2} .

What is the number of 00-free bitstrings of length 30 that have 1 at position 9? (The positions are numbered $1, 2, \dots, 30$.)

- $f_7 \cdot f_{20}$
- $f_8 \cdot f_{21}$
- $f_9 \cdot f_{22}$
- $f_{10} \cdot f_{23}$
- None of the other answers is correct

Question 12 (b). A string over the alphabet $\{x, y, z\}$ is called *fabulous* if it does not contain xyz , xyx , or xx . For $n \geq 1$, let A_n denote the number of fabulous strings of length n . Which of the following is true for any $n \geq 4$?

- $A_n = A_{n-1} + A_{n-2} + A_{n-3}$
- $A_n = 2A_{n-1} + A_{n-2} + A_{n-3}$
- $A_n = 2A_{n-1} + 2A_{n-2} + A_{n-3}$
- $A_n = 2A_{n-1} + 2A_{n-2} + 2A_{n-3}$
- None of the other answers is correct

Question 13 (b). Consider the following recursive function $\text{BAR}(n)$:

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BAR(n) :  
  if  $n \leq 3$  then  
    return  $n$   
  return  $\text{BAR}(n - 1) + \text{BAR}(n - 3)$ 
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When running $\text{BAR}(46)$ how many calls are there to $\text{FU}(41)$?

- 4
- 6
- 8
- 9
- 10

Question 14 (b). You are given an infinite supply of red marbles and an infinite supply of blue marbles. Let S_n be the number of ways of placing n of these marbles in a line so that you never have three red marbles in a row. Which of the following is true, for any $n \geq 3$?

- $S_n = 2S_{n-1}$
- $S_n = 4S_{n-3}$
- $S_n = S_{n-1} + S_{n-2} + S_{n-3}$
- $S_n = S_{n-1} + 2S_{n-2} + S_{n-3}$
- $S_n = 2^n$

Question 15 (b). You toss a fair coin 12 times. Define the event:

$A =$ “the results of the last three flips are equal”

What is $\text{Pr}(A)$?

- $1/2$
- $1/4$
- $1/6$
- $1/8$
- $1/16$

Question 16 (b). A bag contains r red balls, b blue balls, and g green balls. We reach into the bag and choose a uniformly random subset of 2 balls. Define the event:

$A =$ “this subset has no blue balls”

What is $\Pr(A)$?

- $1/\binom{r+g+b}{2}$
- $rg/\binom{r+g+b}{2}$
- $\binom{r+g}{2}/\binom{r+g+b}{2}$
- $(r+g)^2/(r+g+b)^2$
- rg/rgb

Question 17 (b). A bag contains r red balls, b blue balls, and g green balls. We reach into the bag and choose a uniformly random subset of 2 balls. Define the event:

$A =$ “the subset contains one red ball and one green ball”

What is $\Pr(A)$?

- $1/\binom{r+g+b}{2}$
- $rg/\binom{r+g+b}{2}$
- $\binom{r+g}{2}/\binom{r+g+b}{2}$
- $(r+g)^2/(r+g+b)^2$
- rg/rgb