COMP 2804 — Assignment 3

Question 1:

- Write your name and student number.

Topics to explore:

1. Probability and Uniform Probability
   (a) Complement Rule
   (b) Sum Rule
   (c) Inclusion/Exclusion
2. Newton-Pepys
3. Birthday Paradox
4. Conditional Probability
5. Are they independent?
6. Independent Events

Question 2: Dwayne Jetski is a famous hockey player who has a powerful (though sometimes wild) slap shot. Any time he shoots the puck on goal, he scores with probability $\frac{1}{6}$ and the puck goes into the crowd with probability $\frac{1}{3}$.

a) In a typical night, Dwayne has 10 shots on goal. What is the probability that Dwayne scores at least 1 goal?

b) A hat trick is where a player gets 3 goals in a night. What is the probability that Dwayne scores a hat trick (that is he scores *at least* 3 goals)?

c) Over a stretch of 10 games Dwayne takes 50 shots. What is the probability that Dwayne scores exactly 10 goals or exactly 10 shots go into the crowd?

Question 3: There are 100 students enrolling in computer science in Paradox University. Each student must take 5 courses as follows:

- There are 6 first year computer science courses - each student must choose 2 of them.
- For each comp sci course there are 2 tutorials and each student must choose 1.
- There are 15 electives. Among these electives are 7 humanities courses. Each student must choose 3 electives, and they must take at least one humanities (though they may take up to 3 humanities courses if they wish).

A schedule is *unique* if it is different from every other schedule in at least one course or tutorial.

a) How many unique schedules are there?

b) None of the students know what courses to take, so they all choose uniformly at random from the set of unique schedules. What is the probability that all students have a unique schedule?

c) What is the probability that exactly 2 students share the same schedule, but everyone else has a unique schedule?

d) What is the probability that 2 or 3 students share the same schedule, but everyone else has a unique schedule?

Question 4: Bayes' Theorem involves deriving reasonable guesses at probabilities based on statistics, and then updating those probabilities as you get more information. We will use the statistics below and Bayes' Theorem to derive an initial guess at certain probabilities. *Bayes' Theorem* states that:

$$\Pr(A|B) = \frac{\Pr(B|A) \cdot \Pr(A)}{\Pr(B)}.$$

In addition to Bayes' Theorem, you might find the Law of Total Probability useful:

$$\Pr(A) = \Pr(A|B) \cdot \Pr(B) + \Pr(A|\overline{B}) \cdot \Pr(\overline{B}).$$

Out of 1000 people total that took COMP2804 last year, 820 passed the final exam.[1]. 800 students studied for the final exam. 60 people who did not study still passed the final exam. Use these numbers to define initial probabilities and answer the following questions.

a) What is your probability of passing the final exam if you study?

b) Prove that $\Pr(A|B) + \Pr(\overline{A}|B) = 1$.

c) You know someone who failed the final. What is the probability that they studied?

Question 5: You roll 5 fair 6-sided dice. Let C be the event that there are exactly 3 dice that are showing the same number. Let D be the event that there is at least one number i, $1 \leq i \leq 6$, such that exactly 2 of the 6 dice are showing i.

[1] These numbers are completely made up. If you do not study, you will not pass the final.

a) What is $\Pr(C)$?

b) What is $\Pr(D)$?

c) What is $\Pr(C \cup D)$?

d) Are events C and D independent? In other words, is $\Pr(C \cap D) = \Pr(C) \cdot \Pr(D)$?

Question 6: When playing poker you use a standard deck of cards consists of 52 cards. Each card consists of a rank chosen from 13 available ranks and a suit chosen from 4 available suits. The ranks are, in order from least to greatest, $\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$. The suits are $\{\diamondsuit, \spadesuit, \heartsuit, \clubsuit\}$. All suits are considered equal value. Note that there are 4 cards of any given rank. For example, all cards of rank 7 would be $\{7\diamondsuit, 7\spadesuit, 7\heartsuit, 7\clubsuit\}$. In poker you are dealt a hand of 5 cards.

a) What is the probability that you have 4 of a kind? That is, what is the probability that you have 4 cards of the same rank?

b) What is the probability that the highest card in your hand is a 7? We will consider A to be the highest rank overall, thus $A > 7$ and there are 5 ranks below 7.

c) Given that your highest card is a 7, what is the probability that you have four 7's?

d) Given that your highest card is a 7, what is the probability that you have a full house? A full house is three cards of one rank and two cards of another rank.

Question 7: You are on a plane with Samuel L. Jackson when suddenly a crate of 100 snakes opens up. Sam Jackson shouts at the snakes, startling them. As a result each snake bites the tail of another snake, possibly their own. Each snake bites exactly one tail, and each tail is bitten exactly once. Each possible outcome of the snakes biting one another has uniform probability. For this question it may be useful to number the snakes 1..100.

a) How many possible outcomes are there?

b) What is the probability that each snake bites their own tail?

c) What is the probability that all 100 of the snakes form a cycle?

Hint: To count the number of ways all snakes could form a cycle, consider the following sequence of tasks:

Task 1: Snake 1 bites the tail of some snake numbered i, $2 \leq i \leq n$ (since it does not bite its own tail).

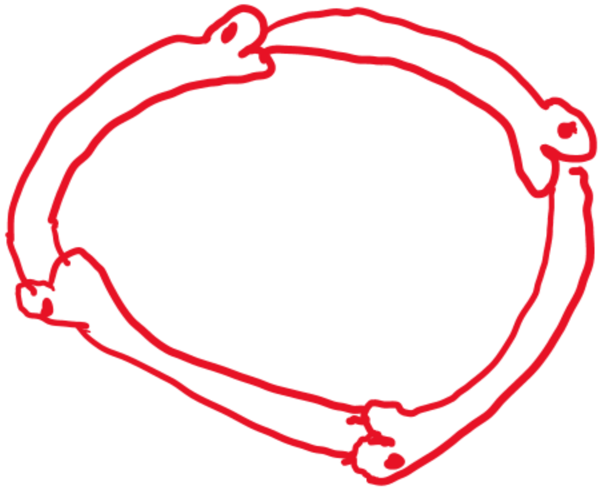

Figure 1: Artist's rendering of a cycle of four snakes. Sam Jackson not pictured.

Task 2: Snake i then bites the tail snake j. We know that $j \neq i$ since i's tail has already been bitten, and $j \neq 1$, since biting the tail of snake 1 would complete a cycle of 3 snakes.

...

Task 100: The last snake bites the tail of snake 1, completing the cycle.

d) Prove that the probability that there are > 50 snakes in the largest cycle is

$$\mathrm{H}_{100} - \mathrm{H}_{50} \approx 0.69$$

where H_i is the i^{th} Harmonic number.