Isoperimetric Enclosures

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Abstract

Given a number P, we study the following three isoperimetric problems introduced by Besicovitch in 1952: (1) Let S be a set of n points in the plane. Among all the curves with perimeter P that enclose S, what is the curve that encloses the maximum area? (2) Let Q be a convex polygon with n vertices. Among all the curves with perimeter P contained in Q, what is the curve that encloses the maximum area? (3) Let $r_\circ$ be a positive number. Among all the curves with perimeter P and circumradius $r_\circ$, what is the curve that encloses the maximum area?

In this paper, we provide a complete characterization for the solutions to Problems 1, 2 and 3. Moreover, we show that there are cases where the solution to Problem 1 cannot be computed exactly, however it is possible to compute in $O(n \log n)$ time the exact combinatorial structure of the solution, leaving only one real parameter, which can be approximated with arbitrary precision. To solve Problem 2, we provide an $O(n \log n)$ time algorithm to compute its solution exactly. In the case of Problem 3, we show that the problem can be solved in constant time.

As a side note, we show that if S is a set of n points in the plane, then finding the curve of perimeter P that encloses S and has minimum area is NP-hard.

1. Introduction

Geometric optimization typically involves finding an object that maximizes or minimizes an objective function subject to geometric constraints. For example, given a set S of points in $\mathbb{R}^2$ we can ask for the minimum enclosing circle of S [14].

In this paper, we study optimization problems involving simple planar curves of fixed perimeter, in other words involving *isoperimetric curves*. In general, *isoperimetric problems* involve optimizing a given function over a family of isoperimetric curves. The classical isoperimetric problem is to maximize the area enclosed by a curve of fixed perimeter. The ancient Greeks knew that the solution to this problem is the circle. This result is known as the *Isoperimetric Theorem*. However, a rigorous proof was not obtained until the late 19th century. The main issue was to prove the existence of this curve as the limit of a sequence of polygons that approximate the circle [6]. An intuitive proof of the Isoperimetric Theorem can be found in the book of Polya [15]. A dual theorem states that among all plane figures of equal area, the circle is the one with minimum perimeter.

Dido's problem, which is present in Greek mythology, is closely related. According to legend, Dido was fleeing from her homeland and seeking asylum in northern Africa. She was offered a piece of land as large as she could encompass with an oxhide. After cutting the hide into one long strip, she formed a curve between two coastal points, thus claiming a large area that later came to be the city of Carthage. Dido's problem is hence known as that of maximizing the area bounded by a straight line (in her case, the coast) and a curve of fixed length. As Dido figured out, the maximum area is obtained when the curve is in the shape of a half circle with endpoints on the coast line.

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Some variants of isoperimetric problems have been addressed by adding geometric constraints [5, 8, 9, 13, 15]. Bose and De Carufel [8] considered the family of isoperimetric triangles enclosing a set S of n points, with the additional constraint that one angle is also fixed. They provided an algorithm to find one such triangle of maximum (or minimum) area in $O(n^2)$ time and showed how to solve a dual version of the problem. That is, among all triangles sharing the same area that enclose S and have one equal angle, compute a triangle with maximum (or minimum) perimeter. Koutsoupias et al. [13] asked how to bisect a simple polygon into two equal (possibly disconnected) parts with the smallest total perimeter. They showed that if the polygon is convex, then the problem can be solved in quadratic time. Otherwise, the problem is NP-complete. Besicovitch [5] studied the following three problems without considering their algorithmic aspects.

1. Among all the convex curves with perimeter P containing a given convex region, what is the curve that encloses the maximum area?
2. Among all the convex curves with perimeter P contained in a given bounded convex region, what is the curve that encloses the maximum area?
3. Among all the convex curves with perimeter P and a given circumradius $r_\circ$, what is the curve that encloses the maximum area?

He characterized the solutions to these problems and provided detailed proofs for the last two. He suggested that similar proofs should work for the first problem. In this paper, we focus on the algorithmic aspects of these three problems. We study the following particular instances of the first two problems and address the third in its full extent.

Problem 1. *Let S be a set of n points in the plane and let $P > 0$ be a given value. Among all the curves with perimeter P that enclose S, what is the curve that encloses the maximum area?*

Problem 2. *Let Q be a convex polygon with n vertices. Among all the curves with perimeter P contained in Q, what is the curve that encloses the maximum area?*

Problem 3. *Among all the convex curves with perimeter P and a given circumradius $r_\circ$, what is the curve that encloses the maximum area?*

In Section 2, we provide a full geometric characterization (with complete proofs) of the solution to Problem 1. In line with what Besicovitch [5] stated, we show that the solution to this problem is a convex curve consisting of circular arcs of equal radius. Therefore, to obtain a full description of this solution, we need to compute their common radius. However, as we explain in Section 2.2, it is unlikely that a closed-form expression to represent this radius always exists. That is, we show that if *Schanuel's conjecture* [3, Chapter 12] is true, there are instances of Problem 1 for which such a closed-form expression does not exist. However, according to Chow [10], at present, a proof of this conjecture seems to be out of reach. In light of this result, in Section 2.1, we present an algorithm to compute an approximation of the solution to Problem 1 together with the exact combinatorial structure of the optimal curve. We show how to find a curve σ made of circular arcs that encloses S in $O(n \log n)$ time such that its perimeter is arbitrarily close to P, and its enclosed area is maximum among all curves that share the same perimeter. Moreover, if $\mathscr{C}$ is the exact solution to Problem 1, then the arcs on σ share their endpoints with the arcs on $\mathscr{C}$ and just one parameter, say the radius of the arcs on $\mathscr{C}$, has no known closed-form expression and is hence approximated.

Besicovitch [5] proved that the optimal solution to Problem 2 is either a circle, or a convex curve that consists of an alternating sequence of circular arcs with equal radii and *sub-edges* (line segments included in the edges) of Q. In Section 3, we show that using the medial axis of Q we can compute a closed-from expression for the common radius of these circular arcs. Using this result, we explain how to obtain the combinatorial structure of the optimal solution. This results in an $O(n \log n)$-time algorithm to compute an exact optimal solution to Problem 2. Thus, even though Problems 1 and 2 are similar, the complexities of computing their solutions are very different.

In Section 4, we address Problem 3. Besicovitch [5] proved that the optimal solution to Problem 3 is a convex curve made of two symmetric circular arcs —a *lens*— such that both endpoints belong to the circumcircle of the lens. Consequently, we can solve Problem 3 in $O(1)$ time and, provided a mild assumption on the model of computation, we can represent the optimal solution by a closed-form expression.

It is worth noting that the problems presented in this paper have dual formulations where all instances of "perimeter" (*resp.* "maximum") are swapped by "area" (*resp.* "minimum"). For example, the dual of Problem 1 asks, among all curves of a given area A that enclose S, what is the curve that has the minimum perimeter? For each of the problems presented in this paper, the solution to their dual is analogous to the solution of their primal formulation.

In Section 5, we consider a similar problem: Among all the curves with perimeter $P > 0$ that enclose a set S of n points, what is the curve that encloses the minimum area? We show that this problem is NP-hard and leave the characterization of its solution as an open question.

2. Isoperimetric curves enclosing points (Problem 1)

We say that a closed curve is *convex* if the region it encloses is convex. The *area* of a closed curve is the area it encloses. Given a positive number P, a P*-curve* is either a simple closed curve with perimeter P or a simple open curve of length P. Given a subset R of the plane, a (P,R)*-curve* is a P-curve that encloses R. Therefore, if $\mathscr{C}$ is a (P,R)-curve, then P is at least as large as the perimeter of the convex hull of R. It is easy to prove that a curve with perimeter P and maximum area must be convex (for instance, refer to [6, p.531]). The following lemma is a simple extension of that statement. We provide a proof for the sake of completeness.

Lemma 1. *Given a number $P > 0$ and a set of points S, any (P,S)-curve of maximum area is convex.*

Proof. Let $\mathscr{C}$ be a (P,S)-curve of maximum area. Assume that $\mathscr{C}$ is not convex for the sake of contradiction. Therefore, there exists a line ℓ passing through two points x and y on $\mathscr{C}$ such that the open segment (x,y) is contained in the exterior of $\mathscr{C}$, and $\mathscr{C}$ is completely contained in one of the closed halfplanes defined by ℓ.

Let $\gamma_{x,y}$ be the open curve along $\mathscr{C}$ that joins x with y and that is contained in the interior of the convex hull of $\mathscr{C}$. Let $\gamma^*_{x,y}$ be the reflection of $\gamma_{x,y}$ with respect to the line ℓ. By replacing $\gamma_{x,y}$ by $\gamma^*_{x,y}$ on $\mathscr{C}$, we obtain a curve $\mathscr{C}^*$ with perimeter P but with larger area. Moreover, the region enclosed by $\mathscr{C}$ is also enclosed by $\mathscr{C}^*$. Because $\mathscr{C}$ and $\gamma^*_{x,y}$ lie on opposite sides of ℓ, $\mathscr{C}^*$ is simple and hence it is a (P,S)-curve with larger area than $\mathscr{C}$, yielding a contradiction. Therefore, $\mathscr{C}$ must enclose a convex region of the plane. □

Since any convex curve enclosing a point set also encloses its convex hull, for the rest of the paper, we assume that we are given a convex n-gon Q that we want to enclose with some convex (P,Q)-curve. This assumption adds a $\Theta(n \log n)$ preprocessing step to the final algorithm.

Given two open curves $\mathscr{C}_1$ and $\mathscr{C}_2$ sharing at least one endpoint, $\mathscr{C}_1 + \mathscr{C}_2$ denotes the curve obtained by the concatenation of the two curves. The following lemma is a direct consequence of the Isoperimetric Theorem, however, for the sake of completeness we include a full proof.

Lemma 2. *Let xy be a closed segment and let P be a positive number. If $\mathscr{C}$ is an open P-curve joining x with y such that $\mathscr{C} + xy$ is a simple closed curve of maximum area, then $\mathscr{C}$ is a circular arc.*

Proof. Assume that $\mathscr{C}$ is not a circular arc for the sake of contradiction. Let D be the unique circular arc with endpoints x and y of length P (modulo reflection). Let D^o be the circular arc such that $D + D^o$ is a circle and let α be the length of D^o. By definition, $\mathscr{C}$ is the P-curve such that $\mathscr{C} + xy$ is of maximum area. Thus, $\mathscr{C} + D^o$ is a $(P+\alpha)$-curve having an area at least as large as that of $D + D^o$ since the cap formed by D^0 and xy is present on both curves. However, by the Isoperimetric Theorem proved in Chapter X of [15], we know that $D + D^0$ is the unique $(P+\alpha)$-curve of maximum area, which yields a contradiction. Thus, $\mathscr{C}$ is a circular arc equal to D. □

Given a closed region R of the plane, let ∂R denote its boundary.

Lemma 3. *Let P be a positive number and let Q be a convex polygon. If $\mathscr{C}$ is a (P,Q)-curve of maximum area, then $\mathscr{C}$ is a sequence of one or more curves, each one being either a circular arc or a segment flush with ∂Q. Moreover, if $\mathscr{C}$ is not a circle, then the endpoints of these curves lie on ∂Q.*

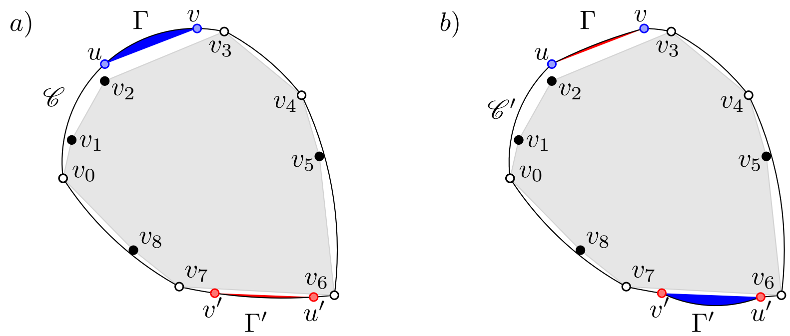

Figure 1: Illustration of the proof of Theorem 4. Polygon $Q = v_0v_1...v_8$ appears in light gray. a) Two arcs Γ and Γ' of $\mathscr{C}$ having different radii. Points u, v and u', v' lie on the arcs Γ and Γ', respectively, with the property that $|uv| = |u'v'|$. b) By swapping the arc that connects u with v along Γ with the one that connects u' with v' along Γ', we obtain a non-convex curve $\mathscr{C}'$ with the same area and perimeter.

Proof. Let z be a point on $\mathscr{C}$. Because $\mathscr{C}$ encloses Q, z lies either in the complement of Q or on the boundary of Q. If z lies in the complement of Q, then let z^* be the closest point to z in Q and let ℓ be the perpendicular bisector of the segment zz^*. Let Π_ℓ be the open halfplane supported by ℓ that contains z. By Lemma 1, $\mathscr{C}$ is convex. Therefore, ℓ intersects $\mathscr{C}$ at exactly two points, x and y. Let $\gamma_z \subset \mathscr{C}$ be the curve joining x with y contained in Π_ℓ. By Lemma 2, γ_z must be a circular arc whose interior contains z. Therefore, every point of $\mathscr{C}$ is either on the interior of a circular arc if it lies in the complement of Q, or otherwise lies on the boundary of Q. □

Lemma 3 implies that no two arcs of $\mathscr{C}$, supported by different circles, can meet in the complement of Q, i.e., every arc of $\mathscr{C}$ must have its endpoints on the boundary of Q. That is, the curve $\mathscr{C}$ is the concatenation of circular arcs and segments flush with the boundary of Q. We now prove the main result of this section.

Theorem 4. *Let P be a positive number and let Q be a convex polygon. If $\mathscr{C}$ is the (P, Q)-curve of maximum area, then $\mathscr{C}$ is a convex curve that consists of a sequence of circular arcs with equal radii. Moreover, if $\mathscr{C}$ consists of at least two circular arcs, then the endpoints of all these circular arcs are vertices of Q.*

Proof. By Lemma 3 we know that $\mathscr{C}$ is a convex P-curve that consists of circular arcs, some of which can have an infinite radius. Suppose that at least two arcs of $\mathscr{C}$ have different radii (possibly infinite) for the sake of contradiction. Let Γ and Γ' be two such arcs; see Figure 1. Let $u, v \in \Gamma$ and $u', v' \in \Gamma'$ be points such that straight-line segments uv and $u'v'$ have equal length. Moreover, by taking this length sufficiently small, we can ensure that the segments uv and $u'v'$ do not intersect the interior of Q.

Let γ be the circular arc with endpoints u and v contained in Γ. Define γ' analogously for u', v' and Γ'. By swapping γ and γ', we obtain a (P, Q)-curve $\mathscr{C}'$ such that area$(\mathscr{C})$ = area$(\mathscr{C}')$. However, since the radii of Γ and Γ' are different, $\mathscr{C}'$ is not convex. Thus, $\mathscr{C}'$ is not optimal by Lemma 1 and hence, there is a curve with perimeter P and area larger than A, which contradicts the optimality of $\mathscr{C}$. Therefore, all circular arcs of $\mathscr{C}$ must have the same radius.

By Lemma 3, we know that every arc of $\mathscr{C}$ has its two endpoints on the boundary of Q. However, if two arcs of $\mathscr{C}$ meet at a point on ∂Q that is not a vertex of Q, we would obtain a non-convex curve, yielding a contradiction to Lemma 1. Therefore, if two arcs of $\mathscr{C}$ meet, they do so at a vertex of Q. □

Let $\mathcal{B}$ be the smallest disk that contains Q. Notice that if the given perimeter P is larger than the perimeter of $\mathcal{B}$, then the (P, Q)-curve of maximum area is a circle with perimeter P that encloses Q (see Chapter X of [15]). Therefore, from now on we assume that P is smaller than the perimeter of $\mathcal{B}$. We observe the following.

Lemma 5. *Given a number P smaller than the perimeter of $\mathcal{B}$, let $\mathscr{C} = (v_0, v_1, \ldots, v_k, v_0)$ be the (P, Q)-curve of maximum area such that v_i and v_{i+1} are connected by a circular arc a_i that intersects Q only at its*

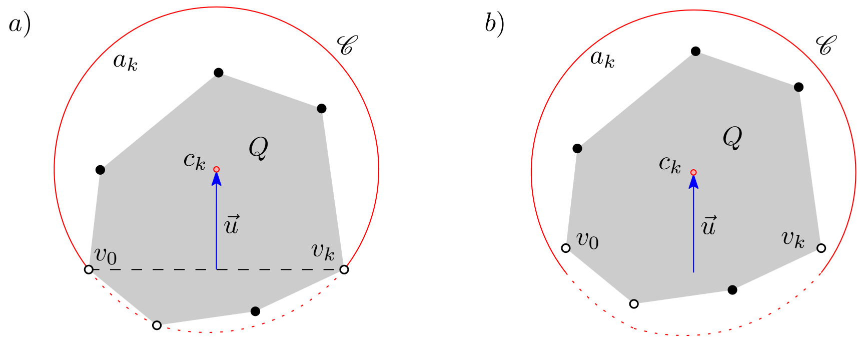

Figure 2: a) A (P, Q)-curve $\mathscr{C}$ with an arc longer than half a circle with radius r. b) The motion of Q in the direction of $\vec{u}$ results in Q being contained in the interior of $\mathscr{C}$, while having no point on its boundary.

endpoints[5]*. If r is the radius of every arc along $\mathscr{C}$, then each of these arcs is at most as long as half a circle with radius r.*

Proof. Assume for a contradiction that $\mathscr{C}$ has an arc, say a_k, longer than half a circle with radius r. Let c_k be the center of the circle that extends a_k. Recall that v_k and v_0 define the endpoints of a_k and let $\vec{u}$ be the vector going from the orthogonal projection of c_k on the segment v_0v_k to c_k. Note that the norm of $\vec{u}$ is strictly greater than zero; see Figure 2(a).

Recall that $\mathscr{C}$ is a convex curve that encloses Q by Lemma 1. Since a_k intersects Q only at its endpoints and because a_k is longer than half a circle with radius r, we can slightly move Q in the direction of $\vec{u}$ in such a way that Q is still enclosed by $\mathscr{C}$ and completely contained in its interior; see Figure 2(b). That is, no point of Q lies on $\mathscr{C}$ after this motion. Hence, $\mathscr{C}$ is still a (P, Q)-curve made of at least two arcs whose endpoints are no longer vertices of Q. By Theorem 4, we infer that $\mathscr{C}$ is not optimal, which is a contradiction. Therefore, every arc along $\mathscr{C}$ is at most as long as half a circle with radius r. □

To complete the characterization of the (P, Q)-curve of maximum area, we use the following result.

Lemma 6. *(Cauchy's Arm Lemma [17, p. 110]) Let $Q = (v_0, v_1, \ldots, v_k, v_0)$ be a convex polygon where each consecutive pair of vertices is connected by an edge $e_i = v_{i-1}v_i$ for $1 \leq i \leq k+1$. The internal angle at vertex v_i between e_i and e_{i+1} is θ_i for $0 \leq i \leq k$. If we remove the edge v_kv_0 from Q and increase the value of some nonempty subset of the angles θ_i while keeping the length of all remaining edges fixed and every $\theta_i \leq \pi$, then the distance between the endpoints v_0 and v_k strictly increases.*

Lemma 7. *Let $\mathscr{C} = (v_0, v_1, \ldots, v_k, v_0)$ be the (P, Q)-curve of maximum area, where v_i and v_{i+1} are connected by a circular arc a_i. If C_i is the circle extending the arc a_i, then Q is enclosed by C_i.*

Proof. If $\mathscr{C}$ is a circle, then it consists of a unique arc and the result follows. Therefore, assume that $\mathscr{C}$ consists of at least two circular arcs. We prove the result for C_0, however, the proof is the same for every circle extending an arc of $\mathscr{C}$.

We claim that for every $0 \leq j \leq k$, the vertex v_j lies inside or on the boundary of C_0. If this claim is true, then by Lemma 5 and the fact that the curvature of C_0 and every arc along $\mathscr{C}$ is the same, $\mathscr{C}$ is contained in C_0. Moreover, because Q is enclosed by $\mathscr{C}$, we conclude that Q is also enclosed by C_0, yielding our result.

Let c_0 be the center of C_0. Let ℓ_0 (*resp.* ℓ_1) be the line passing through c_0 and v_0 (*resp.* c_0 and v_1). Notice that ℓ_0 and ℓ_1 split the plane into four regions R_0, R_1, R_2 and R_3, labeled in clockwise order around c_0, starting with the region containing the arc a_0; see Figure 3(a). To prove that v_j lies inside or on the boundary of C_0, we consider two cases depending on the position of c_0.

[5]Throughout this paper, all index manipulation is modulo $k+1$.

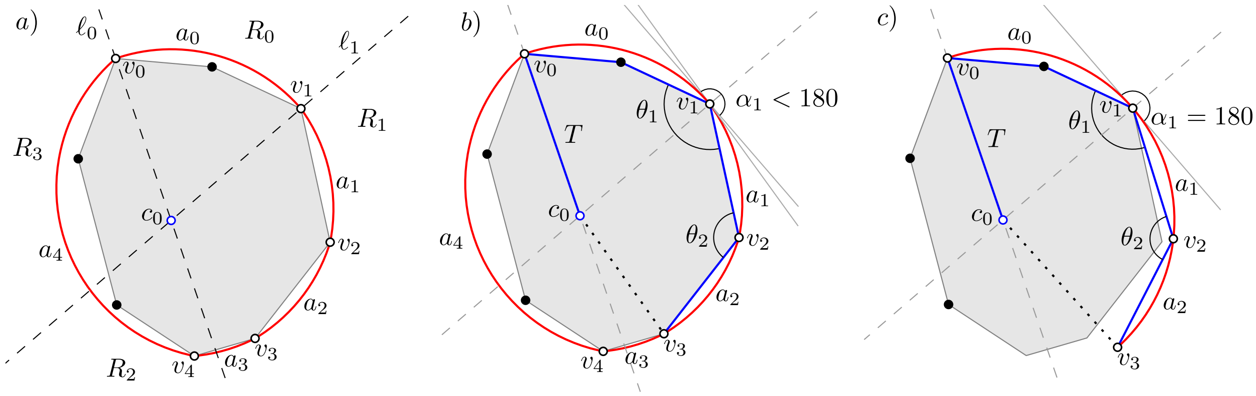

Figure 3: a) The four regions defined by the lines through v_0 and c_0, and through v_1 and c_0, where c_0 is the center of the circle extending the arc a_0. b) The construction to prove that v_3 is in the circle extending a_0. The path $T = (c_0, v_0, v_1, v_2, v_3)$ and the angles $\alpha_1, \alpha_2, \theta_1$ and θ_2 are depicted, all being smaller than π. c) By opening α_1 and α_2 until $\alpha_1 = \alpha_2 = \pi$, v_0, v_1 and v_2 become co-circular while the angles θ_1 and θ_3 increase. The points c_0 and v_3 get farther away after this deformation. Thus, v_3 was inside the circle extending a_0 before the deformation of T.

Case 1. The point c_0 lies inside of Q. In this case, assume that v_j lies in R_1. A similar proof follows if v_j lies in R_2 or R_3. Consider the simple polygon $T = (c_0, v_0, v_1, \ldots, v_{j-1}, v_j, c_0)$ enclosed by $\mathscr{C}$. Since every vertex of $\mathscr{C}$ is a vertex of Q by Theorem 4 and the fact that $v_j \in R_1$, we know that T is a convex polygon. For $0 < i < j$, consider the two lines passing through v_i that are tangent to C_i and C_{i-1}, respectively, and let α_i be the external angle between these lines; see Figure 3(b) for an illustration. Because $\mathscr{C}$ is convex, every α_i is at most π and equality holds if and only if both a_{i-1} and a_i belong to the same circle.

Remove the edge v_jc_0 from T to obtain a polygonal chain T° with endpoints c_0 and v_j. Continuously deform T° by making every angle α_i equal to π while keeping the length of its edges fixed, i.e., we make $v_0, v_1, \ldots, v_j$ co-circular while maintaining the distance between consecutive vertices along T°. If we assume that c_0, v_0 and v_1 remain at their original location, then every vertex v_i of T° ends up lying on the circle C_0 after this deformation. By increasing the value of each α_i until $\alpha_i = \pi$, the internal angle $\theta_i = \angle v_{i-1}v_iv_{i+1}$ at v_i also increases while remaining smaller than π; see Figure 3(c). Therefore, Lemma 6 guarantees that the distance between the endpoints of T° increases after this deformation, i.e., the points c_0 and v_j get farther apart. Because every vertex of T° lies on the boundary of C_0 after the deformation, v_j was originally closer to c_0 and hence, it was enclosed by circle C_0. An analogous proof follows if v_j lies in R_3 by considering the convex polygon $T = (c_0, v_j, v_{j+1}, \ldots, v_k, v_0, v_1, c_0)$. If v_j lies in R_2, let x be the intersection between the segment v_0v_1 and the line through v_j and c_0. Then, consider the convex polygon $T = (c_0, x, v_1, v_2, \ldots, v_{j-1}, v_j, c_0)$ and apply the same argument.

Case 2. If c_0 lies outside of Q, then no vertex of Q lies in R_2. Notice that if v_j lies in R_0, it is contained in the triangle $\triangle v_0v_1c_0$ which implies by convexity that v_j is enclosed by C_0. Assume that v_j lies in R_1 and notice that the same proof used in Case 1 holds as long as $T = (c_0, v_0, v_1, \ldots, v_j, c_0)$ defines a convex polygon. If T is not convex, let $1 \leq i \leq j$ be the largest index such that $T' = (c_0, v_0, v_1, \ldots, v_i, c_0)$ is convex. Therefore, v_i lies inside circle C_0 using the proof of Case 1. Because v_j lies in R_1, v_j lies inside the triangle $\triangle c_0, v_0, v_i$. Moreover, as c_0, v_0 and v_i are all enclosed by C_0, so is v_j by convexity; see Figure 4. An analogous proof holds if v_j lies in R_3, yielding our result. □

Given a number $r \geq radius(\mathcal{B})$, let $\mathscr{D}_r$ be the set of disks with radius r that contain Q. Let φ_r be the intersection of all the disks in $\mathscr{D}_r$.

Lemma 8. *Let $\mathscr{C}$ be the (P, Q)-curve of maximum area and let r be the radius of every arc along $\mathscr{C}$. If $P < perimeter(\mathcal{B})$, then the (P, Q)-curve $\mathscr{C}$ is the boundary of φ_r.*

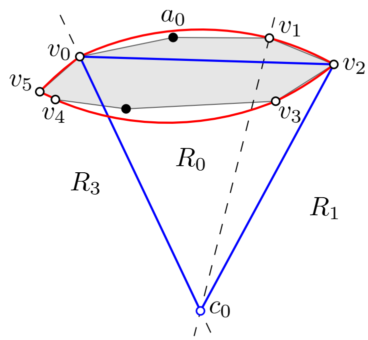

Figure 4: Proof of Case 2 of Lemma 7 where v_2 is the vertex with the largest index i such that the curve $T' = (c_0, v_0, v_1, \ldots, v_i, c_0)$ is convex.

Proof. Since Q is contained in every $D \in \mathscr{D}_r$, Q is contained in φ_r. Moreover, every arc on the boundary of φ_r has radius r and φ_r is convex since it is the intersection of convex shapes.

Let a be an arc of $\mathscr{C}$ and let D_a be the disk with radius r extending it. Notice that $D_a \in \mathscr{D}_r$ by Lemma 7. We claim that the arc a belongs to the boundary of φ_r. For the sake of contradiction, assume that there is a point x on a such that x is not part of the boundary of φ_r. That is, there exists a disk $D \in \mathscr{D}_r$ such that x lies on a but is not contained in D. By Lemma 5 and the fact that D and D_a have the same radius, one of the endpoints of a lies in the complement of D. Therefore, D does not contain Q as both endpoints of a are vertices of Q by Theorem 4. However, Q is contained in every disk of $\mathscr{D}_r$, which is a contradiction. Consequently, every arc along $\mathscr{C}$ is contained on the boundary of φ_r. □

By Lemma 8, to describe $\mathscr{C}$ we only need to consider the intersection of all disks with radius r that contain Q. However, we further simplify this description using the following result.

Lemma 9. *The intersection of every disk with radius r that contains Q is the intersection of every disk with radius r that contains Q and whose boundary passes through at least 2 vertices of Q.*

Proof. Let B be a disk containing Q. If ∂B passes through no vertex of Q, then we can continuously move B to the left (*resp.* right), until ∂B reaches a vertex of Q, to obtain a disk B^- (*resp.* B^+). Since $Q \subset B^+ \cap B^- \subset B$, we can ignore all disks whose boundary contains no vertex of Q when describing φ_r.

If the boundary of B passes through a single vertex v of Q, we can rotate B clockwise or counterclockwise around v until ∂B reaches a second vertex of Q. In this way, we obtain two disks B^- and B^+ whose intersection is a lune contained in B. Because $Q \in B^+ \cap B^-$, by ignoring B, the intersection of all disks with radius r that contain Q remains unchanged. In other words, we can consider only the disks that pass through at least two vertices of Q to describe φ_r. □

2.1. Computing the (P,Q)-curve of maximum area

In this section, we show how to compute the (P,Q)-curve of maximum area using the farthest-point Voronoi diagram of the vertices of Q.

The farthest-point Voronoi diagram of the vertices of Q, denoted by $\mathcal{V}(Q)$, is a geometric tree with n unbounded edges. This tree decomposes the plane into convex regions, each being the locus of points that are farther from a vertex v of Q than from any other vertex of Q [11]. We can think of the leaves of this tree as points at infinity in the direction of its unbounded edges.

Let $c_{\mathcal{B}}$ be the center of $\mathcal{B}$ and let $r_{\mathcal{B}}$ be its radius. Because $c_{\mathcal{B}}$ lies either on a vertex of $\mathcal{V}(Q)$ or on one of its edges, we can assume $\mathcal{V}(Q)$ to be rooted at $c_{\mathcal{B}}$ (if $c_{\mathcal{B}}$ is not a vertex, insert it by splitting the edge where it belongs). Given a point x in the plane, let $\rho(x)$ be the radius of the minimum enclosing circle of Q centered on x. The following lemma states a well-known property of the farthest-point Voronoi diagram.

Lemma 10. *(Proposition 1 of [4]) The map ρ is monotonically increasing along any shortest path from $c_{\mathcal{B}}$ to a leaf of $\mathcal{V}(Q)$.*

Given a number $r \geq r_{\mathcal{B}}$, let $X_r = \{x \in \mathbb{R}^2 : \rho(x) = r$ and x is a point on $\mathcal{V}(Q)\}$.

Lemma 11. *Let $\mathscr{C}$ be the (P, Q)-curve of maximum area and let r be the radius of every arc along $\mathscr{C}$. If $P <$ perimeter($\mathcal{B}$), then $\mathscr{C}$ is the boundary of the intersection of every disk with radius r centered on a point of X_r.*

Proof. Let D be a disk with radius r that contains Q such that ∂D passes through two vertices v and v' of Q. Since $P <$ perimeter($\mathcal{B}$), by Lemma 8 and Lemma 9, it suffices to prove that the center of D belongs to X_r. Let c be the center of D and notice that any disk with radius $r' < r$ centered on c does not contain v and v' and hence, it does not contain Q. Moreover, as D contains Q, we conclude that $\rho(c) = r$. Because ∂D passes through v and v', c is equidistant from these vertices. Furthermore, as Q is contained in D, there cannot be a vertex of Q that is farther from c than v (or v'). Therefore, c lies on the boundary of the farthest-point Voronoi cells of both v and v'. That is, c must lie on an edge of $\mathcal{V}(Q)$, which implies that $c \in X_r$. □

By Lemma 7, the circle extending every arc on $\mathscr{C}$ contains Q. Therefore, the radius of every arc along $\mathscr{C}$ must be greater than the radius of $\mathcal{B}$. Recall that for any value $r \geq r_{\mathcal{B}}$, φ_r is the intersection of all disks with radius r that contain Q.

Lemma 12. *Given any radius $r > r_{\mathcal{B}}$, φ_r and its perimeter can be obtained in $O(n)$ time after computing the farthest-point Voronoi diagram of the vertices of Q.*

Proof. Because $r > r_{\mathcal{B}}$, Lemma 10 implies that on every path joining the root with a leaf of $\mathcal{V}(Q)$, there is a point x such that $\rho(x) = r$.

By Lemma 10, we can scan every edge of $\mathcal{V}(Q)$ to find those edges that contain a point of X_r. Because each of these edges represents the bisector of two vertices of Q, we can determine the position of all the points of X_r in $O(n)$ time. Furthermore, once X_r is computed, we can reconstruct their cyclic order along the boundary of φ_r by performing an Eulerian tour around the tree $\mathcal{V}(Q)$ in linear time. Thus, φ_r and its perimeter can be computed in $O(n)$ time since all circular arcs have the same curvature. □

We finish this section by providing an algorithm to compute an approximation of the (P, Q)-curve of maximum area with fixed but arbitrary precision. Notice that if we are not given a convex polygon but a set of n points, we need to compute their convex hull in $\Theta(n \log n)$ time as a preprocessing step.

Theorem 13. *Let Q be a convex polygon and let $\mathcal{B}$ be the smallest disk containing Q. Given a value $P > 0$, it holds that:* (1) *If $P \geq$ perimeter($\mathcal{B}$), then the (P, Q)-curve $\mathscr{C}$ of maximum area can be computed in $O(n)$ time.* (2) *If perimeter(Q) $< P <$ perimeter($\mathcal{B}$), then we can compute an approximation of the (P, Q)-curve $\mathscr{C}$ of maximum area with arbitrary but fixed precision in $O(n \log n)$ time. Moreover, this approximation has the same combinatorial structure as the optimal solution.*

Proof. Compute the smallest disk $\mathcal{B}$ containing Q in $O(n)$ time [14, 18]. Two cases arise:
(1) If $P \geq$ perimeter($\mathcal{B}$), let c be the center of $\mathcal{B}$. The (P, Q)-curve of maximum area is the circle with perimeter P centered on c.
(2) If $P <$ perimeter($\mathcal{B}$), then compute the farthest-point Voronoi diagram $\mathcal{V}(Q)$ of the vertices of Q in $O(n)$ time [1]. Let r be the radius of every arc in $\mathscr{C}$. Because $\mathscr{C}$ is equal to the boundary of φ_r by Lemma 11, we can use Lemma 12 to approximate r. First, consider every vertex v of $\mathcal{V}(Q)$ and assume that $\rho(v)$ was stored during the computation of $\mathcal{V}(Q)$.

Sort the vertices of $\mathcal{V}(Q)$ by their value under ρ in $O(n \log n)$ time. Then, we can approximate the radius of the arcs of $\mathscr{C}$ using a binary search for r in the set $\{\rho(v) : v$ is a vertex of $\mathcal{V}(Q)\}$. That is, for a given r' in this set, compute $\varphi_{r'}$ and its perimeter in $O(n)$ time using Lemma 12. Then, compare perimeter($\varphi_{r'}$) with P: If this perimeter is larger than P, then $r > r'$; otherwise, $r \leq r'$. This way, we will find two vertices u and v of $\mathcal{V}(Q)$ such that $\rho(u) < r < \rho(v)$. Moreover, we know that for any value $\rho(u) < r' < \rho(v)$, the curve $\varphi_{r'}$ has the same combinatorial structure, i.e., the vertices of $\varphi_{r'}$ and their order along its boundary are the same. Since $\rho(u) < r < \rho(v)$, we have a combinatorial description of $\mathscr{C}$, where only the radius r of the arcs along $\mathscr{C}$ has not yet been determined.

By Theorem 18, there are instances where we cannot compute r exactly. However, we can use binary search on the interval $[\rho(u), \rho(v)]$ to approximate it. That is, for a given constant $\varepsilon > 0$, we can compute an approximation r' of r such that $|r' - r| < \varepsilon$ in $O(n \log \frac{1}{\varepsilon})$ time. Therefore, we obtain a curve that approximates $\mathscr{C}$ with arbitrary precision and that has the same combinatorial structure. □

2.2. Hardness of the computation of exact solutions

In this section, we prove that Problem 1 cannot always be solved exactly. By an *exact solution*, we mean a solution that can be represented by a *closed-form expression*. The problem with closed-form expressions is that there is no consensus in the literature on how they should be defined [7]. According to Borwein and Crandall [7], closed-form expressions can be considered to be "a topic that intrinsically has no 'right' answer." For different reasons that we explain in the next paragraphs, we adopt the definition of Chow [10], qualified by Borwein and Crandall [7] as "the smallest plausible class of closed forms." We first make a little detour to discuss polynomial equations. Let

$$p_d(x) = 0 \tag{1}$$

be a polynomial equation of degree d. If $d \leq 4$, there exist general formulas to solve (1). These general formulas are finite (closed-form) expressions that involve $+$, $-$, $\times$, $\div$ and $\sqrt[k]{\ }$ for any integer $k \geq 2$. In this case, we say that (1) is *solvable by radicals*. If $d > 4$, some polynomial equations are solvable by radicals. For instance, $x^5 - 10x^4 + 35x^3 - 50x^2 + 24x = x(x-1)(x-2)(x-3)(x-4)$. However, some polynomial equations, such as $x^5 - x + 1 = 0$ are not solvable, as can be proven using Galois theory.

If the operations available at unit cost in the model of computation are $+$, $-$, $\times$, $\div$ and $\sqrt[k]{\ }$ for any integer $k \geq 2$, then the solutions to (1) cannot always be computed exactly when $d > 4$. For some problems, the degree of any polynomial equation involved in the solution is bounded by some constant μ. One way of getting around the unsolvability issue is to assume that any polynomial equation of degree less than μ can be solved in $O(1)$ time in the model of computation. However, for some problems, the degree of the polynomial equations involved can be shown to be unbounded in general. In such cases, the solution cannot be computed exactly and we usually turn to approximation algorithms.

Let $\mathscr{C}$ be the (P, Q)-curve of maximum area. Assume that we are given the sequence $v_0, \ldots, v_k$ of vertices of Q that connect consecutive edges along $\mathscr{C}$. That is, $\mathscr{C} = (v_0, v_1, \ldots, v_k, v_0)$ and each v_i is connected to v_{i+1}. By Theorem 4, we know that every arc along $\mathscr{C}$ has the same radius r. Notice that to give an exact description of $\mathscr{C}$, it suffices to compute the value of r.

Let c_i denote the center of the circle extending a_i and notice that c_i lies on the bisector of v_i and v_{i+1}. If d_i denotes half the distance between v_i and v_{i+1}, then the angle $\alpha_i = \angle v_i c_i v_{i+1}$ is equal to $2 \cdot \arcsin(d_i/r)$. Therefore, the perimeter of arc a_i is given by the equation $2r \cdot \arcsin(d_i/r)$. Because the perimeter of $\mathscr{C}$ is P, to find the value of r it suffices to solve the following equation:

$$\sum_{i=0}^{k} 2r \cdot \arcsin(d_i/r) = P \tag{2}$$

This equation is not polynomial and at first glance, it does not seem possible to convert it into a polynomial equation via an appropriate change of variables. We will show that in general, there is no closed-form expression (in Chow's sense [10]) to express the solutions to (2). Following Chow [10], we say that a number can be written in *closed-form* if it is an exponential-logarithmic number.

Definition 14 (Exponential-Logarithmic Numbers, [10]). *Let $\mathbb{E}$ be the set such that $\mathbb{Q} \subset \mathbb{E}$ and for all $x, y \in \mathbb{E}$ with $y \neq 0$, it holds that:*

1. $x + y,\ x - y,\ xy,\ x/y \in \mathbb{E}$,
2. $e^x \in \mathbb{E}$,
3. $\log(y) \in \mathbb{E}$, *where* $\log$ *is the branch of the natural logarithm function such that* $-\pi < Im(\log(y)) \leq \pi$ *for all* y.

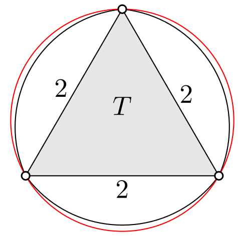

Figure 5: The red curve is the circumcircle of T. The black curve is the solution to Problem 1, when $P = 7$.

Notice that $e \in \mathbb{E}$, $i \in \mathbb{E}$ and $\pi \in \mathbb{E}$, since $e = e^{e^0}$, $i = e^{\log(-1)/2}$ and $\pi = -i\log(-1)$. Therefore, $2\pi i \in \mathbb{E}$ and we have access to all branches of the natural logarithm. We can also compute x^y for any $x, y \in \mathbb{E}$ with $x \neq 0$, since $x^y = e^{y\log(x)}$. Consequently, we can compute the k-th root of any number in $\mathbb{E}$ (for any integer $k \geq 2$). This implies that we can compute the solutions to any polynomial equation with rational coefficients that is solvable by radicals. Finally, we have access to all trigonometric functions, since for instance,

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \;\; \arcsin(x) = -i\log\left(ix + \sqrt{1-x^2}\right), \;\; \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

If we suppose that the operators available at unit cost are $+$, $-$, $\times$, $\div$, $\exp(\cdot)$ and $\log(\cdot)$, we can compute any so-called elementary functions on $\mathbb{E}$ in $O(1)$ time. Most problems from computational geometry can be solved using numbers from $\mathbb{E}$ and the operators defined on $\mathbb{E}$.

Asking whether a polynomial equation can be solved exactly corresponds to asking whether it can be solved by radicals. Asking whether a transcendental equation like $x + e^x = 0$, $\cos(x) = x$ or (2) can be solved exactly corresponds (in Chow's sense [10]) to asking whether its solutions belong to $\mathbb{E}$. Standard results in this matter are of the form: if *Schanuel's conjecture* is true, then the solution to $x + e^x = 0$ does not belong to $\mathbb{E}$ [10]. Here is the statement of Schanuel's conjecture [3, Chapter 12].

Conjecture 15 (Schanuel)**.** *If α_1, α_2,..., α_n are complex numbers linearly independent over $\mathbb{Q}$, then the transcendence degree of the field $\mathbb{Q}(\alpha_1, e^{\alpha_1}, \alpha_2, e^{\alpha_2}, ..., \alpha_n, e^{\alpha_n})$ over $\mathbb{Q}$ is at least n.*

According to Chow [10], at present, a proof of Schanuel's conjecture seems to be out of reach.

We now take an instance of Problem 1 and show that its solution does not belong to $\mathbb{E}$, provided that Schanuel's conjecture is true. Let T be an equilateral triangle such that each side has length 2; see Figure 5. Suppose we want to enclose T with a closed maximum-area curve with perimeter 7. To compute the optimal solution, we need to solve $6r \cdot \arcsin(1/r) = 7$ for $r \in \mathbb{R}$. Equivalently, we need to solve

$$\sin(r') = \frac{6}{7}r', \tag{3}$$

where $r' = \frac{7}{6r}$. Let r' be the solution[6] to (3) such that $r' > 0$. Is $r' \in \mathbb{E}$? Using an approach similar to that of Chow for the proof of Theorem 2 in [10], we can prove the following lemma.

Lemma 16. *If Schanuel's conjecture is true, then $r' \notin \mathbb{E}$.*

To prove Lemma 16, we need the Lindemann-Weierstrass Theorem (see [3, Theorem 1.4]).

Theorem 17 (Lindemann-Weierstrass)**.** *For any distinct algebraic numbers $\alpha_1, \alpha_2, ..., \alpha_n$ and any non-zero algebraic numbers $\beta_1, \beta_2, ..., \beta_n$, we have $\beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \cdots + \beta_n e^{\alpha_n} \neq 0$.*

Proof of Lemma 16. We proceed as in the proof of Theorem 2 in [10]. First notice that $r' \notin \mathbb{Q}$. Indeed, $0 = \sin(r') - 6r'/7 = e^{ir'}/(2i) - e^{-ir'}/(2i) - 6r'e^0/7$, from which we can apply Theorem 17. Because our

[6]We have $r' \approx 0.94683$, from which $r \approx 1.23219$.

focus is on (3), to get in line with the proof of Theorem 2 in [10], Equation (3.2) in this proof needs to be rewritten, which can be done using $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$. The rest of the proof is identical to that of Theorem 2 in [10]. □

As mentioned earlier, in the case of polynomial equations, one can avoid solvability questions by adding an appropriate operator to the model of computation. However, this is reasonable provided that the polynomial equations are of bounded degree. Looking at (3), one could argue that it suffices to add an appropriate operator to the model of computation. We could add an operator that solves equations of the form $\sin(x) = \lambda x$ in $O(1)$ time, where λ can be any real number. However, we claim that in general, the complexity of (2) is unbounded. As for the case of polynomial equations, it means that we need to turn to approximate solutions. To prove our claim we use the following identity.

$$\arcsin(\alpha) + \arcsin(\beta) = \arcsin\left(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}\right) \tag{4}$$

Using (4) and the change of variables $r^* = \frac{P}{2r}$, if all the d_i's are different, (2) becomes $\sin(r^*) = \mathcal{R}(r^*)$, where $\mathcal{R}(r^*)$ satisfies the following: (a) It contains at least one sequence of $k-1$ nested square roots. (b) Each square root in this sequence is summed to a polynomial in r^* of degree at least 2. (c) The most inner radicand is a polynomial in r^* of degree at least 2. Therefore, the complexity of (2) is unbounded and r^* cannot be represented as a closed-form expression. We obtain the following result.

Theorem 18. *If Schanuel's conjecture is true, there exists a convex polygon Q and a value P such that the (P,Q)-curve of maximum area cannot be computed exactly, i.e., the radius of each arc along this curve cannot be represented as a closed-form expression.*

We conclude this section with the following remark. Chow proved [10, Corollary 1] that if Schanuel's conjecture is true, then the polynomial equations with rational coefficients that can be solved within $\mathbb{E}$ are precisely those that are solvable by radicals. This is another reason to define a closed-form as in Definition 14.

3. Isoperimetric curves on an island (Problem 2)

Let Q be a convex polygon with n vertices. Assume for ease of description that Q has no two parallel edges. At the end of this section, we provide further insight for the case when Q has parallel edges. Given a positive number P, a $[P,Q]$-curve is a curve with perimeter P contained in Q. In this section, we provide an algorithm to solve Problem 2. To this end, we begin by characterizing a $[P,Q]$-curve of maximum area.

Let $\mathcal{I}$ be the largest circle contained in Q. Notice that if $P \leq \text{perimeter}(\mathcal{I})$, any disk of perimeter P contained in $\mathcal{I}$ is a solution. Therefore, for the rest of this section, suppose that $P > \text{perimeter}(\mathcal{I})$. The following two results were proved by Besicovitch [5].

Theorem 19. *Let Q be a convex polygon and $P >$ perimeter$(\mathcal{I})$ be a positive number. If $\mathscr{I}$ is a $[P,Q]$-curve of maximum area, then $\mathscr{I}$ is a convex curve described by an alternating sequence $s_0, a_0, s_1, a_1, \ldots, s_k, a_k$, where (1) s_i is a straight-line segment contained in an edge e_i of Q, (2) a_i is a circular arc connecting s_i with s_{i+1} such that the circle extending a_i is contained in Q and is tangent to both e_i and e_{i+1}, and (3) all a_i have the same radius.*

Let ϕ_r be the convex hull of every disk of radius r contained in Q. Since we assumed that Q has no parallel edges, we have the following lemma (refer to [5]).

Lemma 20. *Let $\mathscr{I}$ be a $[P,Q]$-curve of maximum area and let r be the radius of every arc in $\mathscr{I} \setminus \partial Q$. If perimeter$(\mathcal{I}) < P <$ perimeter(Q), then $\mathscr{I}$ is the boundary of ϕ_r.*

By Lemma 20, since we assumed that Q has no parallel edges, the solution to Problem 2 is unique and we can describe it by looking at all the disks of radius r contained in Q. However, we further simplify this description using the following result.

Lemma 21. *The convex hull ϕ_r of every disk of radius r contained in Q is the convex hull of every disk of radius r contained in Q and whose boundary is tangent to at least two edges of Q.*

Proof. Let B be a disk with radius r contained in Q. If no edge of Q is tangent to B, we can continuously move B to the left (*resp.* right), until B becomes tangent to an edge of Q, to obtain a disk B^- (*resp.* B^+). Since $B \subset CH(B^- \cup B^+)$, we can ignore all disks that are not tangent to the boundary of Q.

If the boundary of B is tangent to a single edge e of Q, we can translate B parallel to e (in both directions) until B becomes tangent to a second edge of Q. In this way, we obtain two disks B_- and B_+ whose convex hull contains B. Because $B \subset CH(B_- \cup B_+) \subset Q$, by ignoring B, the convex hull of all disks with radius r that are contained in Q remains unchanged. In other words, we can consider only the disks tangent to at least two edges of Q to describe ϕ_r. □

3.1. Computing the $[P, Q]$-curve of maximum area

The medial axis of Q, denoted by $\mathcal{A}(Q)$, can be seen as a tree that splits Q into n regions, each being the locus of points that are closer to an edge e of Q than to any other edge of Q. Moreover, the leaves of this tree are the vertices of Q. Because Q is convex, every edge of $\mathcal{A}(Q)$ is a straight-line segment. The *bisector* of two edges of Q is the locus of points that are equidistant to these edges. Thus, each edge of $A(Q)$ is contained in the bisector of two edges of Q (see [16] for a detailed analysis of this structure).

In this section, we show how to compute the $[P, Q]$-curve of maximum area by using the medial axis of Q. Recall that $\mathcal{I}$ is the largest circle contained in Q and let $c_{\mathcal{I}}$ and $r_{\mathcal{I}}$ denote its center and radius, respectively. Notice that $c_{\mathcal{I}}$ is either a vertex of $\mathcal{A}(Q)$ or lies on one of its edges.

Given a point $x \in Q$, let $\sigma(x)$ be the radius of the largest circle centered on x that is contained in Q. The following is a well-known property of the medial axis of Q.

Lemma 22. *If Q has no parallel edges, then the map σ is strictly decreasing along any shortest path from $c_{\mathcal{I}}$ to a leaf of $\mathcal{A}(Q)$.*

Given a number $r \leq r_{\mathcal{I}}$, let $Y_r = \{x \in Q : \sigma(x) = r \text{ and } x \text{ is a point on } \mathcal{A}(Q)\}$. By Lemma 21, ϕ_r is the convex hull of every disk of radius r centered on a point of Y_r. Because we assumed that Q has no parallel edges, Y_r is a finite set of points.

Lemma 23. *Let $\mathscr{I}$ be the $[P, Q]$-curve of maximum area and let r^* be the radius of every arc in $\mathscr{I} \setminus \partial Q$. If perimeter($\mathcal{I}$) $< P <$ perimeter(Q), then $\mathscr{I}$ is the boundary of the convex hull of every disk of radius r^* centered on a point of Y_{r^*}.*

Proof. Let D be a disk with radius r^* contained in Q such that D is tangent to two edges e and e' of Q. Because perimeter($\mathcal{I}$) $< P <$ perimeter(Q), by Lemmas 20 and 21, it suffices to prove that the center of D belongs to Y_{r^*}. Let c be the center of D. Note that any disk of radius $r' > r^*$ centered on c is not contained in Q. Moreover, because D is contained in Q, we conclude that $\sigma(c) = r^*$. Since D is tangent to e and e', c is equidistant from these edges. Because D is contained in Q, we conclude that c lies on the medial axis of Q and hence, that $c \in Y_{r^*}$. □

Lemma 24. *Given any radius $r > r_{\mathcal{I}}$, after computing the medial axis of Q we can compute ϕ_r and its perimeter in $O(n)$ time.*

Proof. By Lemma 22 and from the fact that $r > r_{\mathcal{I}}$, on every path joining $c_{\mathcal{I}}$ with a leaf of $\mathcal{A}(Q)$, there is a point x such that $\sigma(x) = r$. Therefore, we can scan every edge of $\mathcal{A}(Q)$ to find those edges containing a point of Y_r. Because each edge of $\mathcal{A}(Q)$ is contained in the bisector of two edges of Q and from the fact that $\mathcal{A}(Q)$ has linear size, we can determine the position of all points in Y_r in $O(n)$ time.

For each $y \in Y_r$, mark the edges of Q that are tangent to the circle of radius r centered on y. To construct the boundary of ϕ_r, visit the marked edges of Q in clockwise order around its boundary and connect them with the appropriate arcs contained in the circles that marked them. Thus, ϕ_r and its perimeter can be computed in $O(n)$ time. □

Let $\mathscr{I}$ be a $[P, Q]$-curve of maximum area. Note that if $\mathscr{E} = e_0, \ldots, e_k$ is the sequence of edges of Q visited by the $\mathscr{I}$ in clockwise order, then this sequence encodes the combinatorial structure of $\mathscr{I}$ by Theorem 19. We call $\mathscr{E}$ the *combinatorial sequence* of $\mathscr{I}$.

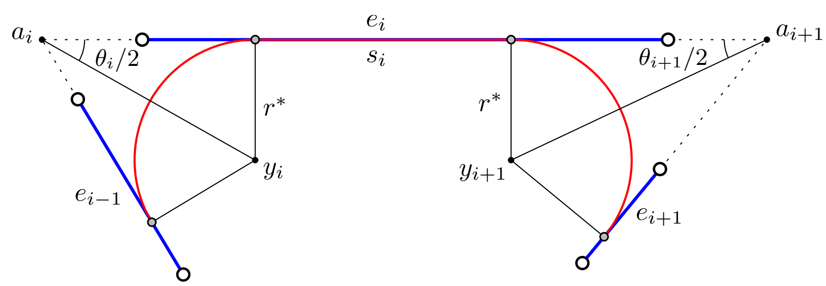

Figure 6: Part of the optimal solution to Problem 2.

Lemma 25. *Let $\mathscr{I}$ be the $[P,Q]$-curve of maximum area. If the combinatorial sequence $\mathscr{E} = \{e_0, e_1, \ldots, e_k\}$ of $\mathscr{I}$ is known, then we can compute the radius r^* of very arc in $\mathscr{I} \setminus \partial Q$ in $O(n)$ time.*

Proof. Recall that by Lemma 23, $\mathscr{I}$ is the boundary of the convex hull of every circle of radius r^* centered on a point of Y_{r^*}. Assume that $Y_{r^*} = \{y_0, \ldots, y_k\}$ is sorted in such a way that y_i lies on the bisector of e_{i-1} and e_i. Let a_i be the point of intersection of the lines extending e_{i-1} and e_i, and let θ_i be the angle that these lines make in the wedge with apex a_i that contains Q; see Figure 6 for an illustration.

Note that the length of the circular arc of $\mathscr{I}$ contained in the circle of radius r^* centered on y_i has length $(\pi - \theta_i)r^*$. Recall that by Theorem 19, $\mathscr{I}$ consists of an alternating sequence of arcs and segments flushed with ∂Q. To account for these segments let s_i denote the segment of $\mathscr{I}$ contained in e_i (s_i could be a single point if $\mathscr{I}$ is tangent to e_i). In this case, the length of s_i is given by

$$|a_i a_{i+1}| - \frac{r^*}{\tan(\theta_i/2)} - \frac{r^*}{\tan(\theta_{i+1}/2)} = |a_i a_{i+1}| - \left(\frac{1}{\tan(\theta_i/2)} + \frac{1}{\tan(\theta_{i+1}/2)}\right) r^*.$$

Since e_i and e_{i+1} are fixed, the values of $|a_i a_{i+1}|$ and $\frac{1}{\tan(\theta_i/2)} + \frac{1}{\tan(\theta_{i+1}/2)}$ can be computed exactly in a preprocessing step. Then, the perimeter of ϕ_{r^*} is given by the following formula

$$\text{perimeter}(\phi_{r^*}) = \sum_{i=0}^{k} \left[(\pi - \theta_i) r^* + |a_i a_{i+1}| - \frac{r^*}{\tan(\theta_i/2)} - \frac{r^*}{\tan(\theta_{i+1}/2)} \right].$$

Thus, to find the value of r^*, we need to solve the equation perimeter$(\phi_{r^*}) = P$ which depends only on $\mathscr{E}$. Because this equation is linear in r^*, we can solve it exactly in $O(n)$ time. ☐

We conclude this section by providing an algorithm to compute a $[P,Q]$-curve of maximum area.

Theorem 26. *Let Q be a convex polygon and let $\mathcal{I}$ be the largest disk contained in Q. Given a value $P > 0$, it holds that: (1) If perimeter$(\mathcal{I}) \geq P$, a $[P,Q]$-curve of maximum area can be computed in $O(n)$ time. (2) If perimeter$(\mathcal{I}) < P <$ perimeter(Q), the $[P,Q]$-curve of maximum area can be computed in $O(n \log n)$ time.*

Proof. Compute the largest disk contained in Q in $O(n)$ time [2]. Two cases can arise. (1) If perimeter$(\mathcal{I}) \geq P$, let $c_{\mathcal{I}}$ be the center of $\mathcal{I}$. A $[P,Q]$-curve of maximum area is the circle with perimeter P centered on $c_{\mathcal{I}}$. (2) If perimeter$(\mathcal{I}) < P <$ perimeter(Q), then compute the medial axis $\mathcal{A}(Q)$ in $O(n)$ time [2]. Let $\mathscr{I}$ denote the $[P,Q]$-curve of maximum area. Let r^* denote the radius of every arc in $\mathscr{I} \setminus \partial Q$. Our goal is to find the combinatorial sequence $\mathscr{E}$ of $\mathscr{I}$ and use it in conjunction with Lemma 25 to compute r^*.

Sort the vertices of $\mathcal{A}(Q)$ by their values under the map σ in $O(n \log n)$ time. Then, use binary search on the vertex set of $\mathcal{A}(Q)$ to find two vertices u and u' such that $\sigma(u) < r^* < \sigma(u')$. To this end, fix a vertex v of $\mathcal{A}(Q)$ and compute $\phi_{\sigma(v)}$ in $O(n)$ time using Lemma 24. Then, compare the perimeter of $\phi_{\sigma(v)}$ with P. If they are equal, then $\mathscr{I}$ is the boundary of $\phi_{\sigma(v)}$ and we are done. If the perimeter of $\phi_{\sigma(v)}$ is larger than P, then $r^* > \sigma(v)$. Otherwise, if this perimeter is smaller than P, then $r^* < \sigma(v)$. Because $\mathcal{A}(Q)$ has $O(n)$ vertices and from the fact that computing $\phi_{\sigma(v)}$ takes $O(n)$ time, we can find two vertices u and u' such that

$\sigma(u) < r^* < \sigma(u')$ in $O(n \log n)$ time. At this point, for any $\sigma(u) < r' < \sigma(u')$ (including r^*), the boundary of $\phi_{r'}$ has the same combinatorial structure.

To compute the combinatorial sequence $\mathscr{E}$ of $\mathscr{I}$, let A be the set of edges of $\mathcal{A}(Q)$ that contain a point whose value under σ lies between $\sigma(u)$ and $\sigma(u')$. That is, each edge of A contains exactly one point of Y_{r^*}. By traversing each edge of $\mathcal{A}(Q)$ and comparing the value of its endpoints under σ with $\sigma(u)$ and $\sigma(u')$, we can compute A in $O(n)$ time by Lemma 22. For each $a \in A$, mark the two edges of Q whose bisector defines a. Then, walk along the boundary of Q in clockwise order and whenever a marked edge is visited, append this marked edge of Q to $\mathscr{E}$. In this way, we obtain the combinatorial sequence $\mathscr{E}$ of $\mathscr{I}$.

Once $\mathscr{E}$ is determined, we can use Lemma 25 to compute r^* in $O(n)$ time. Finally, we use Lemma 24 to compute $\mathscr{I}$. Thus, if perimeter($\mathcal{I}$) $< P <$ perimeter(Q), then we can compute $\mathscr{I}$ in $O(n \log n)$ time. $\square$

We now explain how to solve the general case where Q can have parallel edges. Note that the approach described above works as long as Y_r is a finite set of points. Otherwise, because Q is convex, the map σ is constant on at most one edge a of $A(Q)$ being the bisector of two parallel edges e and e' of Q. If Y_r is the set of points on a line segment $s \subseteq a$, then $\mathscr{I}$ contains a circular arc that is tangent to e and e'. In this case, the radius r^* of every arc in $\mathscr{I} \setminus \partial Q$ is half the distance between e and e' by Theorem 19. Thus, $\mathscr{I}$ consists of two half-circles of radius r^* and two segments contained in e and e'. Finding the curve with this constraints of perimeter P is a constant size problem and can be solved exactly in $O(1)$ time.

4. Isoperimetric curves restricted to a circumcircle (Problem 3)

Besicovitch [5] proved the following result.

Theorem 27. *The optimal solution to Problem 3 is a convex curve made of two symmetric circular arcs —a lens— such that both endpoints belong to the given circumcircle and the total perimeter is P.*

Therefore, given any instance of Problem 3, the radius r of the optimal lens satisfies the following equation

$$2 \arcsin\left(\frac{r_\circ}{2r}\right) = \frac{P}{2r}, \tag{5}$$

where $r_\circ$ is the radius of the given circumcircle. As we explain in Section 2.2, there is no closed-form expression to represent the solutions to (5). However, (5) is of bounded complexity. Therefore, if we suppose that we can solve the equation $\sin(x) = \lambda x$ in $O(1)$ time, where λ can be any real number, then we can solve Problem 3 exactly in $O(1)$ time. Note that a similar assumption is not enough to solve Problem 1 as there are instances whose complexity is unbounded.

5. Open problems

We conclude by stating open problems that are closely related to the problem presented in this paper.

Problem 4. *Given a set of points $S \in \mathbb{R}^2$ and a value $P > 0$, find the (P, S)-curve of minimum area.*

Even though Problem 4 has a formulation similar to that of Problem 1, it belongs to a different class of problems as stated in the following theorem.

Theorem 28. *Problem 4 is NP-hard.*

Proof. A Steiner tree of S is a geometric tree whose vertex set contains S but may also contain additional vertices. Let $\mathcal{T}$ be a minimum Euclidean Steiner tree of S, i.e., a minimum cost Steiner tree where the cost of an edge is its Euclidean length. Notice that if P is equal to twice the length of $\mathcal{T}$, then the solution to Problem 4 is a closed curve of area zero obtained by going around $\mathcal{T}$. Because computing the minimum Steiner tree is NP-hard [12], computing the solution to Problem 4 is also NP-hard. $\square$

By extending the formulation of isoperimetric problems to $\mathbb{R}^3$, we obtain the following problem statement that, as far as we know, has not yet been studied.

Problem 5. *Let $S \in \mathbb{R}^3$ be a set of points and let $A > 0$. Among all surfaces of area A, what is the closed surface of (maximum) minimum volume that encloses S?*

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